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# Lie–Bäcklund symmetries of dispersionless, magneto-hydrodynamic model equations near the triple umbilic point*

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**Abstract.** Lie–Bäcklund symmetries and conservation laws are derived for weakly nonlinear magnetohydrodynamic (MHD) equations describing the interaction of the Alfvén and magnetoacoustic modes propagating parallel to the ambient magnetic field, in the parameter regime near the triple umbilic point, where the gas sound speed $a_g$ matches the Alfvén speed $V_A$. The dispersive form of the equations can be expressed in Hamiltonian form and admit four Lie point symmetries and conservation laws associated with space-translation invariance (momentum conservation), time translation invariance (energy conservation), rotational invariance about the magnetic field $\boldsymbol{B}$ (helicity conservation), plus a further symmetry that is associated with accelerating wave similarity solutions of the equations. The main aim of the paper is a study of the symmetries and conservation laws of the dispersionless equations. The dispersionless equations are of hydrodynamic type and have three families of characteristics analogous to the slow, intermediate and fast modes of MHD and the Riemann invariants for each of these modes are given in closed form. The dispersionless equations are shown to be semi-Hamiltonian, and to possess two infinite families of symmetries and conservation laws. The analysis emphasizes the role of the Riemann invariants of the dispersionless equations and a hodograph transformation for a restricted version of the equations.

## 1. Introduction

Nonlinear finite amplitude Alfvén waves and magnetosonic waves are of fundamental interest to plasma physics, space plasma physics and astrophysics. In the solar wind the scale lengths of the background plasma flow at sufficiently large distances from the Sun are much greater than the typical Alfvén wavelength, and hence provide a natural laboratory for testing nonlinear theories for Alfvén waves and their coupling to magnetosonic modes, as well as wave generation by electromagnetic instabilities.

Early work by Taniuti and Wei [1] and Kakutani *et al* [2] showed that the propagation of weakly nonlinear, long-wavelength dispersive magnetosonic waves at a finite non-zero angle to the background magnetic field was governed by the Korteweg–deVries (KdV) equation, whereas the Alfvén wave satisfies the modified Korteweg–deVries (mKdV) equation.

The quasi-parallel propagation of MHD waves along the magnetic field is degenerate in the sense that the Alfvén speed $V_A$ then matches one of the magneto-acoustic speeds $V_f$ or $V_s$ for the fast and slow magnetosonic modes. In this case the canonical equation governing the evolution of right-hand and left-hand polarized Alfvén waves is the derivative nonlinear Schrödinger (DNLS) equation [3–5]. The DNLS equation is an integrable Hamiltonian

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system, with an infinite number of conservation laws. It's initial value problem can be solved exactly by the inverse scattering transform (IST), which is the nonlinear analogue of the Fourier transform [6, 7]).

Generalized forms of the DNLS equation for a kinetic guiding centre plasma, including the effects of a finite plasma beta, and a non-local term representing the effects of resonant particles, as well as three dimensional effects associated with wave diffraction have been derived by Mjølhus and Wyller [8]. Related work on modified forms of the DNLS equation appropriate for warm, multi-species plasmas with anisotropic pressures has been carried out by Verheest [9], Khabibrakhmanov *et al* [10], and DeConinck *et al* [11, 12]).

The derivation of the DNLS equation assumes that the Alfvén and sound speeds are well separated and distinct. Work by Brio [13,14], Hada [15], Brio and Rosenau [16], and Passot and Sulem [17] considers the appropriate form of the wave evolution equations for quasi-parallel propagation of the Alfvén and magneto-acoustic modes near the triple umbilic point, where the sound speed and Alfvén speed are almost equal (i.e. $a_{\mathrm{g}}^2/V_{\mathrm{A}}^2 - 1 = \epsilon\Delta$ where $a_{\mathrm{g}}$ is the gas sound speed, $V_{\mathrm{A}}$ is the Alfvén speed, and $\epsilon$ is the perturbation parameter representing the wave amplitude). In this limit, the coefficient of the nonlinear term in the DNLS equation diverges, and a modified version of the method of multiple scales that explicitly takes into account the fact that $a_{\mathrm{g}}^2/V_{\mathrm{A}}^2 - 1$ is a small quantity must be used. The resulting equations have appropriately been described as the triple degenerate DNLS system (the TDNLS system), since the Alfvén, fast magneto-acoustic and slow magneto-acoustic waves have the same phase speed to lowest order in $\epsilon$. The divergence of the nonlinear term in the DNLS equation as $a_{\mathrm{g}}/V_{\mathrm{A}} \to 1$ only occurs in the MHD two fluid model. The coefficient of the nonlinear term for the kinetic guiding centre model is similar to that in the MHD model of the DNLS equation for $T_{\mathrm{e}} \gg T_{\mathrm{i}}$ where $T_{\mathrm{e}}$ and $T_{\mathrm{i}}$ denote the electron and ion temperatures [18], but the two coefficients differ substantially for $T_{\mathrm{i}} \sim T_{\mathrm{e}}$. Studies of the modulational instability of circularly polarized Alfvén waves for the TDNLS equations have been carried out by Hada [15]. Related work on the modulational instability has been carried out by Hollweg [19].

Webb *et al* [20] showed that the TDNLS equations admit both Lagrangian and Hamiltonian variational formulations. The Lie point symmetries admitted by the equations were used to derive classical similarity solutions. The dispersive TDNLS equations possess four Lie point symmetries associated with: time translation invariance, space translation invariance, rotational invariance and a further symmetry. The first three symmetries correspond, via Noether's theorem, to the energy, momentum and helicity conservation laws. An analysis of the prolongation Lie algebra suggested that the dispersive TDNLS system for the case $\gamma_{\mathrm{g}} \neq 0$ is non-integrable, but is possibly an integrable system in the limit $\gamma_{\mathrm{g}} \to 0$, where $\gamma_{\mathrm{g}}$ is the gas adiabatic index.

The main purpose of the present paper is to study the relationship between the symmetries and conservation laws of the dispersionless TDNLS equations. The dispersionless TDNLS system is of hydrodynamic type and has three families of characteristics analogous to the slow, intermediate and fast modes of MHD, and the Riemann invariants for each of these modes can be obtained in closed form [20]. A general theory of the symmetries and conservation laws of equations of hydrodynamic type has been developed by Dubrovin and Novikov [21], Tsarev [22, 23] and others (see, e.g., Ferapontov [24]). Integrability conditions on the characteristic speeds were established by Tsarev [22,23] in order that the system admit an infinite number of commuting flows (or Lie–Bäcklund symmetries) and conservation laws. Application of this theory to the TDNLS system shows that the dispersionless TDNLS equations admit an infinite number of commuting flows and conservation laws.

A brief overview of the dispersive TDNLS system is presented in section 2. In section 3, the characteristic speeds and Riemann invariants of the dispersionless TDNLS equations are established. We also discuss the Hamiltonian and Poisson bracket structure of the dispersionless equations in section 3. In section 4, a search for conservation laws of the dispersionless TDNLS equations of the form

$$\frac{\partial A}{\partial t} + \frac{\partial F}{\partial x} = 0 \tag{1.1}$$

in which the conserved densities $A(u, v, w)$ and fluxes $F(u, v, w)$ depend only on the dependent variables $u$, $v$ and $w$ yields an infinite number of conservation laws. The conserved densities turn out to satisfy a linear, second-order, hyperbolic differential equation. The separated solutions of this latter equation yield infinite families of conserved densities $A$. In section 5, Hamiltonian theory [22–25] is then used to relate the symmetries and conservation laws for the dispersionless TDNLS equations. In section 6, Lie–Bäcklund symmetries of the equations are obtained by exploiting the hodograph transformation for a restricted form of the equations. In section 7, the commutation relations for the symmetry algebra of the non-dispersive TDNLS equations are given. We conclude in section 8 with a summary and discussion.

## 2. The dispersive TDNLS system

The TDNLS system studied by Brio [13, 14], Hada [15], and by Brio and Rosenau [16] describes the quasi-parallel propagation and interaction of the fast, slow and intermediate MHD modes along the background magnetic field in the special limit where the gas sound speed matches to lowest order in the perturbation parameter $\epsilon$, the Alfvén speed $V_{\mathrm{A}}$, and the fast, slow and Alfvén speeds coincide to lowest order. The TDNLS equations are derived from the equations for two fluid, Hall current plasmas by the method of multiple scales. The dimensional form of the TDNLS equations may be written as

$$\frac{\partial}{\partial t}\left(\frac{\delta\rho}{\rho_0}\right) + V_{\mathrm{A}}\frac{\partial}{\partial x}\left[\frac{\gamma_{\mathrm{g}}+1}{4}\left(\frac{\delta\rho}{\rho_0}\right)^2 + \frac{1}{2}\left(\frac{a_{\mathrm{g}}^2}{V_{\mathrm{A}}^2} - 1\right)\frac{\delta\rho}{\rho_0} + \frac{\delta B^+\delta B^-}{4B_0^2}\right] = 0 \tag{2.1}$$

$$\frac{\partial}{\partial t}\left(\frac{\delta B^\pm}{B_0}\right) + V_{\mathrm{A}}\frac{\partial}{\partial x}\left[\frac{1}{2}\frac{\delta\rho}{\rho_0}\frac{\delta B^\pm}{B_0} \pm \mathrm{i}\frac{\chi}{2}\frac{\partial}{\partial x}\left(\frac{\delta B^\pm}{B_0}\right)\right] = 0 \tag{2.2}$$

where

$$\delta B^\pm = \delta B_y \pm \mathrm{i}\delta B_z = \epsilon B_0(v \pm \mathrm{i}w) \qquad \delta\rho = \epsilon\rho_0 u \tag{2.3}$$

represent the complex transverse field perturbations ($\delta B^\pm$), and density perturbations ($\delta\rho$). The density perturbation $\delta\rho$ and $x$-component of the fluid velocity perturbation are related by the eigenequation $\delta u = V_{\mathrm{A}}\delta\rho/\rho_0$. In equations (2.1), (2.2) it is assumed that

$$\frac{a_{\mathrm{g}}^2}{V_{\mathrm{A}}^2} - 1 = \epsilon\Delta \tag{2.4}$$

where $\Delta$ is a constant of order unity. The parameter $\chi = V_{\mathrm{A}}/\Omega_{\mathrm{p}}$ is the ion inertial length and $x = X - V_{\mathrm{A}}t$ denotes position in the wave frame; $\rho_0$ and $B_0$ denote the background density and magnetic field induction and $\gamma_g$ is the adiabatic index of the gas. The dimensionless form of the TDNLS equations used by Brio and Rosenau [16] are

$$\frac{\partial u}{\partial \bar{t}} + \frac{\partial}{\partial \bar{x}}\left(\frac{\Gamma u^2 + v^2 + w^2}{2}\right) = 0 \tag{2.5}$$

$$\frac{\partial \psi}{\partial \bar{t}} + \frac{\partial}{\partial \bar{x}}[(u - \Delta)\psi + \mathrm{i}\bar{\chi}\psi_{\bar{x}}] = 0 \tag{2.6}$$

where $\psi = v + \mathrm{i}w$ represents the complex transverse magnetic field perturbations, $u$ represents the density or $x$-component of the fluid velocity perturbation, and

$$\Gamma = \gamma_g + 1. \tag{2.7}$$

The stretched space and time variables in equations (2.5)–(2.6) $\bar{x}$ and $\bar{t}$ are defined by

$$\bar{x} = \tilde{x} - \Delta\bar{t} \qquad \tilde{x} = \frac{\epsilon x}{\chi} \qquad \bar{t} = \tfrac{1}{2}\epsilon^2\Omega_p t. \tag{2.8}$$

where $\Omega_p$ is the proton gyro-frequency. The parameter $\bar{\chi}$ in equation (2.6) represents Hall current dispersion. In ion inertial units, in which distance is measured in terms of the ion inertial length $V_A/\Omega_p$, and time in terms of the proton gyro-time scale $\Omega_p^{-1}$, the parameter $\bar{\chi} = 1$. Our main concern in the present paper is with the dispersionless form of the TDNLS equations with $\chi = \bar{\chi} = 0$.

In terms of $(u, v, w)$ as dependent variables, the TDNLS system may be expressed in the form

$$u_t + D_x\left(\frac{\Gamma u^2 + v^2 + w^2}{2}\right) = 0 \tag{2.9}$$

$$v_t + D_x[(u - \Delta)v - \chi w_x] = 0 \tag{2.10}$$

$$w_t + D_x[(u - \Delta)w + \chi v_x] = 0 \tag{2.11}$$

where we use the notation $D_x \equiv \partial/\partial x$, and by a convenient abuse of notation we have dropped bars on the normalized quantities $\bar{\chi}$, $\bar{x}$ and $\bar{t}$ which we also use in the following development.

### *2.1. Lie point symmetries and variational formulations*

In this section we briefly note the Lie point symmetries of the dispersive TDNLS system (2.9)–(2.11), and variational formulations of the equations which are described in greater detail in Webb *et al* [20].

The TDNLS equations (2.9)–(2.11) for general $\Gamma$ admit Lie point symmetries of the form

$$\begin{aligned} &x' = x + \epsilon\xi^x \qquad t' = t + \epsilon\xi^t \\ &u' = u + \epsilon\eta^u \qquad v' = v + \epsilon\eta^v \qquad w' = w + \epsilon\eta^w \end{aligned} \tag{2.12}$$

where

$$\begin{aligned} &\xi^x = c_1 - c_4[(\Gamma - 1)x - \Gamma\Delta t] \qquad \xi^t = c_2 - 2(\Gamma - 1)c_4 t \\ &\eta^u = c_4[(\Gamma - 1)u + \Delta] \qquad \eta^v = -c_3 w + c_4 v \qquad \eta^w = c_4 w + c_3 v. \end{aligned} \tag{2.13}$$

The corresponding vector field

$$X = \xi^x \frac{\partial}{\partial x} + \xi^t \frac{\partial}{\partial t} + \eta^u \frac{\partial}{\partial u} + \eta^v \frac{\partial}{\partial v} + \eta^w \frac{\partial}{\partial w} = \sum_{i=1}^{4} c_i X_i \tag{2.14}$$

spans the point Lie algebra, where the basis vector fields $\{X_i : i = 1(1)4\}$, are given by

$$\begin{aligned} X_1 &= \frac{\partial}{\partial x} \qquad X_2 = \frac{\partial}{\partial t} \qquad X_3 = v\frac{\partial}{\partial w} - w\frac{\partial}{\partial v} \equiv \frac{\partial}{\partial \theta} \\ X_4 &= -[(\Gamma-1)x - \Gamma\Delta t]\frac{\partial}{\partial x} - 2(\Gamma-1)t\frac{\partial}{\partial t} + [(\Gamma-1)u + \Delta]\frac{\partial}{\partial u} \\ &\quad + (\Gamma-1)\left(v\frac{\partial}{\partial v} + w\frac{\partial}{\partial w}\right) \end{aligned} \tag{2.15}$$

and the $\{c_i : i = 1(1)4\}$ are arbitrary constants. The non-zero commutators of the point Lie algebra are

$$\begin{aligned} [X_1, X_4] &= -[X_4, X_1] = -(\Gamma-1)X_1 \\ [X_2, X_4] &= -[X_4, X_2] = \Gamma\Delta X_1 - 2(\Gamma-1)X_2. \end{aligned} \tag{2.16}$$

The vector field $X_1$ corresponds to space translation invariance; $X_2$ to time translation invariance; and $X_3$ to rotational invariance of the $(v, w)$ variables (i.e. invariance with respect to translations in $\theta$, where $v = r\cos\theta$, and $w = r\sin\theta$). The symmetry $X_4$ is more difficult to characterize.

The TDNLS equations (2.9)–(2.11) can be obtained by extremizing the variational functional

$$J[U^\alpha] = \iint L \, \mathrm{d}x \, \mathrm{d}t \tag{2.17}$$

where the Lagrange density $L$ is given by

$$L = -\tfrac{1}{2}\left[\tfrac{1}{3}\Gamma U_x^3 + \Psi_x \Psi_x^*(U_x - \Delta) + U_x U_t + \tfrac{1}{2}(\Psi_x \Psi_t^* + \Psi_x^* \Psi_t) + \tfrac{1}{2}\mathrm{i}\chi(\Psi_x^* \Psi_{xx} - \Psi_x \Psi_{xx}^*)\right] \tag{2.18}$$

$$u = U_x \qquad \psi = v + \mathrm{i}w = \Psi_x \equiv V_x + \mathrm{i}W_x. \tag{2.19}$$

In equations (2.18) the superscript $*$ denotes complex conjugation and $U$, $V$, $W$ and $\Psi = V + \mathrm{i}W$ are potentials for $u$, $v$, $w$ and $\psi$.

Use of the variational principle (2.17) coupled with Noether's theorem (e.g. [26]), and the Lie point symmetries (2.12), (2.13) yields four conservation laws for the TDNLS equations of the form

$$D_t A_j + D_x F_j = 0 \tag{2.20}$$

where $D_t \equiv \partial/\partial t$ and $D_x \equiv \partial/\partial x$. The conserved densities $\{A_j : j = 1(1)4\}$ and fluxes $\{F_j : j = 1(1)4\}$ corresponding to the symmetry operators $\{X_j : j = 1(1)4\}$ are given below:

$$A_1 = \tfrac{1}{2}(u^2 + v^2 + w^2) \tag{2.21a}$$

$$F_1 = \tfrac{1}{3}\Gamma u^3 + \left(u - \tfrac{1}{2}\Delta\right)(v^2 + w^2) - \chi(vw_x - wv_x) \tag{2.21b}$$

$$A_2 = \tfrac{1}{2}\left(\tfrac{1}{3}\Gamma u^3 + (u-\Delta)(v^2 + w^2) - \chi(vw_x - wv_x)\right) \tag{2.22a}$$

$$F_2 = \tfrac{1}{2}\Big(\tfrac{1}{4}(\Gamma u^2 + v^2 + w^2)^2 + [(u-\Delta)v - \chi w_x]^2 + [(u-\Delta)w + \chi v_x]^2 \\ -\chi(wv_t - vw_t)\Big) \tag{2.22b}$$

$$A_3 = WV_x - VW_x \tag{2.23a}$$

$$F_3 = [(u-\Delta)v - \chi w_x]W - [(u-\Delta)w + \chi v_x]V + \chi(v^2 + w^2) \tag{2.23b}$$

$$A_4 = 2(\Gamma - 1)tA_2 - [(\Gamma-1)x - \Gamma\Delta t]A_1 + \Delta U \tag{2.24a}$$

$$F_4 = 2(\Gamma - 1)tF_2 - [(\Gamma-1)x - \Gamma\Delta t]F_1. \tag{2.24b}$$

Equations (2.21)–(2.24) correspond respectively to the momentum, energy and helicity conservation laws, plus a more obscure conservation law associated with the symmetry operator $X_4$.

The TDNLS system (2.9)–(2.11) can also be written in the Hamiltonian form

$$\begin{aligned} u_t &= D_x\left(\frac{\delta\mathcal{H}}{\delta u}\right) = -D_x\left(\frac{\Gamma u^2 + v^2 + w^2}{2}\right) \\ v_t &= D_x\left(\frac{\delta\mathcal{H}}{\delta v}\right) = -D_x[(u-\Delta)v - \chi w_x] \\ w_t &= D_x\left(\frac{\delta\mathcal{H}}{\delta w}\right) = -D_x[(u-\Delta)w + \chi v_x] \end{aligned} \tag{2.25}$$

where

$$\mathcal{H} = \int_{-\infty}^{\infty} H\,\mathrm{d}x \tag{2.26}$$

$$H \equiv -A_2 = -\tfrac{1}{2}\left(\tfrac{1}{3}\Gamma u^3 + (u-\Delta)(v^2 + w^2) - \chi(vw_x - wv_x)\right) \tag{2.27}$$

defines the Hamiltonian functional $\mathcal{H}$, $D_x$ is the symplectic operator [25] and $\delta\mathcal{H}/\delta u^\alpha$ denotes the variational derivative of $\mathcal{H}$ with respect to $u^\alpha$. The Hamiltonian density $H$ corresponds to the conserved density $A_2$ associated with the time translation symmetry.

The above summary gives the main results in Webb *et al* [20] concerning the dispersive TDNLS system pertinent to the present development. Note that the Lagrangian variational principle (2.17), the Hamiltonian variational principle (2.25), the Lie point symmetries (2.15), and conservation laws (2.21)–(2.24) also apply to the non-dispersive equations in which $\chi = 0$.

## 3. The non-dispersive TDNLS equations

In section 3.1 we establish the characteristic speeds and Riemann invariants of the dispersionless TDNLS system. Using the Riemann invariants as the dependent variables, the equations are reduced to a diagonal system of hydrodynamic type. In section 3.2 we give a brief discussion of the Poisson bracket structure and the geometry of the equations under a transformation of the dependent variables. In particular, the Hamiltonian structure of the equations are discussed for the case where the Riemann invariants are used as the new dependent variables.

### *3.1. The Riemann invariants*

The dispersionless TDNLS system (2.9)–(2.11) with $\chi = 0$ may be written in the matrix form

$$\frac{\partial \boldsymbol{u}}{\partial t} + \mathbf{A} \cdot \frac{\partial \boldsymbol{u}}{\partial x} = 0 \tag{3.1}$$

where $\boldsymbol{u} = (u, v, w)^{\mathrm{T}}$ (the superscript T denotes the transpose) and the matrix $\mathbf{A}$ is given by

$$\mathbf{A} = \begin{pmatrix} \Gamma u & v & w \\ v & u - \Delta & 0 \\ w & 0 & u - \Delta \end{pmatrix}. \tag{3.2}$$

Along a characteristic curve $\mathcal{C}:\ x = x(s),\ t = t(s)$ for equations (3.1)

$$\det(\mathbf{A} - \lambda \mathbf{I}) = 0 \qquad \lambda = \frac{x'(s)}{t'(s)}. \tag{3.3}$$

The Riemann invariants for equations (3.1) are functions $R(u, v, w)$ that are constant along the characteristics, and hence $\nabla_{\boldsymbol{u}} R = (R_u, R_v, R_w)$ is a left eigenvector of the matrix $\mathbf{A}$

$$\nabla_{\boldsymbol{u}} R \cdot (\mathbf{A} - \lambda \mathbf{I}) = 0 \tag{3.4}$$

(see, e.g., [27]).

From equations (3.2) and (3.3), the eigenvalues or characteristic speeds $\lambda$ satisfy the eigenvalue equation

$$\det(\mathbf{A} - \lambda \mathbf{I}) = -[\lambda - (u - \Delta)]\{\lambda^2 + [\Delta - (\Gamma + 1)u]\lambda + \Gamma u(u - \Delta) - r^2\} = 0 \tag{3.5}$$

where $r^2 = v^2 + w^2$. There are three solutions of the eigenvalue equation (3.5), namely

$$\lambda = \lambda_{\mathrm{I}} = u - \Delta \qquad \lambda_{\mathrm{f}} = \lambda_+ = \tfrac{1}{2}[(\Gamma+1)u - \Delta + D^{\frac{1}{2}}] \qquad \lambda_{\mathrm{s}} = \lambda_- = \tfrac{1}{2}[(\Gamma+1)u - \Delta - D^{\frac{1}{2}}] \tag{3.6}$$

where

$$D = [(\Gamma - 1)u + \Delta]^2 + 4r^2. \tag{3.7}$$

In equations (3.6), $\lambda_f > \lambda_I > \lambda_s$, and the subscripts 'f', 'I', and 's' denote the fast, intermediate and slow mode solutions in analogy with the fast, intermediate and slow modes of MHD. From the right eigenvectors of the matrix **A** one can show that the intermediate mode is incompressible, but the fast and slow modes are compressible.

The differential equations (3.4) for the Riemann invariants written in component form are

$$(\Gamma u-\lambda)\frac{\partial R}{\partial u}+v\frac{\partial R}{\partial v}+w\frac{\partial R}{\partial w}=0 \tag{3.8}$$

$$v\frac{\partial R}{\partial u}+(u-\Delta-\lambda)\frac{\partial R}{\partial v}=0 \tag{3.9}$$

$$w\frac{\partial R}{\partial u}+(u-\Delta-\lambda)\frac{\partial R}{\partial w}=0. \tag{3.10}$$

Converting to polar coordinates, $v = r\cos\theta$, $w = r\sin\theta$, equations (3.8)–(3.10) yield the equations

$$(\Gamma u-\lambda)\frac{\partial R}{\partial u}+r\frac{\partial R}{\partial r}=0 \tag{3.11}$$

$$(u-\Delta-\lambda)\frac{\partial R}{\partial \theta}=0. \tag{3.12}$$

For the intermediate mode $\lambda = u - \Delta$, and equations (3.8)–(3.12) yield the Riemann invariant

$$R = R_I = \tan\theta = \frac{w}{v} \tag{3.13}$$

(in fact any pure function of $\theta$ will do as a Riemann invariant).

For the fast and slow modes for which $\lambda \neq (u - \Delta)$, the Riemann invariants $R_f = R_+(u, r)$ and $R_s = R_-(u, r)$ are independent of $\theta$. The Riemann invariants may be obtained by integrating the characteristic equation

$$\frac{du}{dr}=\frac{\Gamma u-\lambda}{r} \tag{3.14}$$

associated with equation (3.11), where the integration constant may be identified with the Riemann invariant. Using the expressions (3.6) for $\lambda_\pm$, the characteristic equation (3.14) may be written in the separable form

$$r\frac{d\mu}{dr}=\frac{1}{2}[(\Gamma-3)\mu-(\Gamma-1)\sigma(\mu^2+4)^{\frac{1}{2}}] \qquad \sigma=\pm 1 \tag{3.15}$$

where

$$\mu=\frac{(\Gamma-1)u+\Delta}{r} \tag{3.16}$$

and the solutions for $\sigma = \pm 1$ correspond to $\lambda = \lambda_\pm$, respectively.

The general solution of the differential equations (3.15) for $\Gamma \neq 1$ and $\Gamma \neq 2$ are

$$R_+ = r|\phi-1|^{-1}|\phi+1|^{\alpha_1}\left|\phi^2-\frac{2(\Gamma-3)}{\Gamma-1}\phi+1\right|^{\alpha_2} \tag{3.17}$$

$$R_- = r|\phi-1|^{\alpha_1}|\phi+1|^{-1}\left|\phi^2+\frac{2(\Gamma-3)}{\Gamma-1}\phi+1\right|^{\alpha_2} \tag{3.18}$$

where

$$\alpha_1 = \frac{1}{\Gamma - 2} \qquad \alpha_2 = \frac{\Gamma - 3}{2(\Gamma - 2)} \qquad \phi = \frac{(4+\mu^2)^{\frac{1}{2}} - 2}{\mu}. \tag{3.19}$$

The integration constants $R_+$ and $R_-$ are the Riemann invariants for the fast and slow modes.

For $\Gamma = 1$, integration of the characteristic equation (3.14) yields the Riemann invariants

$$R_+ = u + \tfrac{1}{2}(g - \Delta \ln|g + \Delta|) \qquad R_- = u - \tfrac{1}{2}(g + \Delta \ln|g - \Delta|) \tag{3.20}$$

where

$$g = (\Delta^2 + 4r^2)^{\frac{1}{2}}. \tag{3.21}$$

For $\Gamma = 2$, integration of equations (3.15) yields the invariants

$$R_+ = r\left|\frac{\phi+1}{\phi-1}\right| \exp\left(\frac{2\phi}{(\phi+1)^2}\right) \qquad R_- = r\left|\frac{\phi-1}{\phi+1}\right| \exp\left(-\frac{2\phi}{(\phi-1)^2}\right) \tag{3.22}$$

where $\phi$ is defined in equation (3.19). For $\Gamma = 3$ ($\gamma_{\rm g} = 2$), the solutions (3.17) and (3.18) simplify to

$$\begin{aligned} R_+ &= r\left|\frac{\phi+1}{\phi-1}\right| = \frac{1}{2}r|\mu + (\mu^2+4)^{\frac{1}{2}}| = |\lambda_+ + \Delta| \\ R_- &= r\left|\frac{\phi-1}{\phi+1}\right| = \frac{1}{2}r|(4+\mu^2)^{\frac{1}{2}} - \mu| = |\lambda_- + \Delta|. \end{aligned} \tag{3.23}$$

Hence $\lambda_+$ is constant on the $C^+$ characteristic and $\lambda_-$ is constant on the $C^-$ characteristic and the characteristics are straight lines in the $(x, t)$ plane for $\gamma_{\rm g} = 2$.

The dispersionless TDNLS system (3.1) when written in terms of the Riemann invariants results in the diagonalized system

$$\frac{\partial R^i}{\partial t} + \lambda^i \frac{\partial R^i}{\partial x} = 0 \qquad i = 1(1)3 \tag{3.24}$$

where $i = 1, 2, 3$ refer to the slow, intermediate and fast modes respectively.

### 3.2. Hamiltonian systems of hydrodynamic type

The dispersionless TDNLS system (3.1) has been reduced to the diagonalized Riemann invariant form (3.24). Both equation systems are clearly Hamiltonian systems, but further analysis is needed to show the Hamiltonian structure of the Riemann invariant form of the equations. The system (3.1) can be written in the Poisson bracket form

$$u^\alpha_t = \{u^\alpha, \mathcal{H}\} = \frac{\partial^2 H}{\partial u^\alpha \partial u^\beta} u^\beta_x \tag{3.25}$$

where $H$ is the Hamiltonian density (2.27) with $\chi = 0$, and $\mathcal{H}$ is the corresponding Hamiltonian functional. In equations (3.25)

$$\{F, G\} = \int_{-\infty}^{\infty} \frac{\delta F}{\delta u^\alpha} D_x\left(\frac{\delta G}{\delta u^\alpha}\right) \mathrm{d}x \tag{3.26}$$

is the Poisson bracket for functionals $F(u^\alpha)$ and $G(u^\alpha)$, with $(u^1, u^2, u^3) = (u, v, w)$. One question addressed in detail by Tsarev [23] is the manner in which the Poisson bracket changes under a general change of variables $w^\alpha = w^\alpha(\boldsymbol{u})$. Of particular interest is the form of the Poisson bracket in terms of the Riemann invariants $R^\alpha = R^\alpha(\boldsymbol{u})$.

For a general change of variables $u^\alpha \to w^\alpha(\boldsymbol{u})$, the Poisson bracket (3.26) assumes the form

$$\{F, G\} = \int_{-\infty}^{\infty} \frac{\delta F}{\delta w^s} A^{sp}\left(\frac{\delta G}{\delta w^p}\right) \mathrm{d}x \tag{3.27}$$

where

$$A^{sp} = \bar{g}^{sp} D_x - \bar{g}^{sa} \bar{\Gamma}^p_{ak} w^k_x \tag{3.28}$$

is the symplectic operator in the new variables and

$$\begin{aligned} \bar{g}^{sp} &= \frac{\partial w^s}{\partial u^\alpha}\frac{\partial w^p}{\partial u^\beta} g^{\alpha\beta} \qquad g^{\alpha\beta} = \delta^\alpha{}_\beta \\ \bar{\Gamma}^a_{bc} &= -\frac{\partial u^\alpha}{\partial w^b}\frac{\partial u^\beta}{\partial w^c}\frac{\partial^2 w^a}{\partial u^\alpha \partial u^\beta} \qquad \Gamma^a_{bc} = 0 \end{aligned} \tag{3.29}$$

defines the metric $\bar{g}^{sp}$ and affine connection $\bar{\Gamma}^a_{bc}$ in the new variables $w^\alpha$, and $\delta^\alpha{}_\beta$ denotes the Kronecker delta symbol. The quantities $g_{ab} = \delta^a{}_b$ and $\Gamma^a_{bc} \equiv 0$, correspond to the metric and affine connection in the $u^\alpha$ variables. The Poisson bracket (3.27) is that used by Dubrovin and Novikov [21] in their work on Hamiltonian systems of hydrodynamic type. From equations (3.27)

$$\{w^a(x, t), w^b(y, t)\} = \bar{g}^{ab}\delta'(x - y) - \bar{g}^{as}\bar{\Gamma}^b_{\theta s} w^\theta_x \delta(x - y) \tag{3.30}$$

is the Poisson bracket for the new variables $\{w^\alpha\}$, where $\delta(x - y)$ denotes the Dirac delta distribution, and $\delta'(x - y)$ is it's derivative. One can show that the Poisson bracket (3.27) is anti-symmetric and satisfies the Jacobi identity if the connection $\bar{\Gamma}^a_{bc}$ is symmetric and if the Riemann curvature tensor $R^\alpha{}_{\beta\gamma\delta}$ is identically zero [23]. In terms of the new variables $\{w^\alpha\}$, the Hamiltonian system (3.25) may be written in the form

$$w^\alpha_t = (\bar{\nabla}^\alpha \bar{\nabla}_\beta H) w^\beta_x = \{w^\alpha, \mathcal{H}\} \tag{3.31}$$

where $\bar{\nabla}_\alpha$ denotes the covariant derivative with associated metric $\bar{g}^{ab}$ and connection $\bar{\Gamma}^a_{bc}$.

One can show that the hydrodynamic type system [23]

$$w^\alpha_t = V^\alpha{}_\beta w^\beta_x \tag{3.32}$$

is Hamiltonian iff the conditions

$$\bar{g}_{ik} V^k{}_j = \bar{g}_{jk} V^k{}_i \qquad \bar{\nabla}_i (V^k{}_j) = \bar{\nabla}_j (V^k{}_i) \tag{3.33}$$

hold. For the special case where $w^\alpha \equiv R^\alpha(\boldsymbol{u})$ are the Riemann invariants, use of the results (3.32), (3.33) imply that the metric $\bar{g}^{ab}$ has the diagonal form

$$\bar{g}^{ab} = \frac{\partial R^a}{\partial u^\alpha}\frac{\partial R^b}{\partial u^\alpha} = \Lambda_a \delta^a{}_b \qquad \bar{g}_{ab} = \frac{\partial u^\alpha}{\partial R^a}\frac{\partial u^\alpha}{\partial R^b} \equiv \frac{1}{\Lambda_a}\delta^a{}_b \tag{3.34}$$

where $\bar{g}^{aa} \equiv \Lambda_a$ denotes the diagonal element of the metric. The connection coefficients in this case are given by

$$\bar{\Gamma}^i_{jk} = 0 \qquad \text{if} \quad i \neq j \neq k$$
$$\bar{\Gamma}^k_{ik} = \bar{\Gamma}^k_{ki} = \partial_i \ln(\bar{g}^{\frac{1}{2}}_{kk}) \equiv \frac{\partial_i \lambda^k}{\lambda^i - \lambda^k} \qquad \text{if} \quad i \neq k \tag{3.35}$$
$$\bar{\Gamma}^k_{kk} = \partial_k \ln(\bar{g}^{\frac{1}{2}}_{kk})$$

and $\partial_i \equiv \partial/\partial R^i$ denotes the partial derivative with respect to the Riemann invariant $R^i$. Finally we note that the equations for $\bar{\Gamma}^k_{ik}$ in equations (3.35) regarded as differential equations for the $\lambda^k$ are integrable if

$$\partial_j\left(\frac{\partial_i \lambda^k}{\lambda^i - \lambda^k}\right) - \partial_i\left(\frac{\partial_j \lambda^k}{\lambda^j - \lambda^k}\right) = 0 \qquad \text{for} \quad i \neq j \neq k. \tag{3.36}$$

Equations (3.36) also turn out to be a requirement that the Riemann invariant equations admit Lie–Bäcklund symmetries.

A diagonal quasilinear system of equations of the form (3.24) in $n$ variables $R^i$ ($n \geqslant 3$) (which is not necessarily Hamiltonian) and for which the $\lambda^i(R)$ are distinct, and that satisfy equations (3.36) is called semi-Hamiltonian (see [23]). These systems admit an infinite number of Lie–Bäcklund symmetries. For $n \leqslant 2$, any system of the form (3.24) is considered semi-Hamiltonian.

In the following two sections we consider conservation laws for the non-dispersive TDNLS system from two perspectives. In section 4 we use the straightforward ansatz (1.1) to derive an infinite family of conservation laws. Then in section 5, we use the theory developed by Tsarev [22, 23] and Ferapontov [24] to relate the conservation laws to commuting flows or Lie–Bäcklund symmetries admitted by the equations.

## 4. Families of conservation laws

A straightforward way in which to derive conservation laws for the dispersionless TDNLS system (3.1) is to search for conserved densities $A(u, v, w)$ and fluxes $F(u, v, w)$ satisfying the conservation equation

$$\frac{\partial A}{\partial t} + \frac{\partial F}{\partial x} = 0. \tag{4.1}$$

This procedure turns out to yield two distinct families of conservation laws. More general conservation laws that depend explicitly on $x$ and $t$, or on the potentials $U$, $V$ and $W$ can clearly be constructed. Examples of conservation laws that are not of this form include the helicity conservation law (equations (2.23)) which depends on the potentials $V$ and $W$, and the conservation law associated with the symmetry operator $X_4$ (equations 2.24) which depends explicitly on $x$ and $t$.

From equations (2.9)–(2.11), the dispersionless TDNLS equations can be written as

$$u_t + \Gamma u u_x + v v_x + w w_x = 0$$
$$v_t + v u_x + (u - \Delta) v_x = 0 \tag{4.2}$$
$$w_t + w u_x + (u - \Delta) w_x = 0.$$

Alternatively using polar coordinates, $v = r\cos\theta$, $w = r\sin\theta$, the equations may be written in the polar coordinate form

$$\begin{aligned} &u_t + \Gamma u u_x + r r_x = 0 \\ &r_t + (u-\Delta) r_x + r u_x = 0 \\ &\theta_t + (u-\Delta)\theta_x = 0. \end{aligned} \tag{4.3}$$

Both forms of the equations will be of use in the following analysis.

Using equations (4.2) in equation (4.1) and equating to zero the coefficients of $u_x$, $v_x$ and $w_x$ yields the equations

$$F_u = \Gamma u A_u + v A_v + w A_w \tag{4.4}$$

$$F_v = v A_u + (u-\Delta) A_v \tag{4.5}$$

$$F_w = w A_u + (u-\Delta) A_w. \tag{4.6}$$

The integrability conditions for equations (4.4)–(4.6) require that $F$ have continuous second-order partial derivatives with respect to $u$, $v$ and $w$. This leads to the compatibility equations

$$2F_{[uv]} = [(\Gamma-1)u+\Delta]A_{uv} + vA_{vv} + wA_{wv} - vA_{uu} = 0 \tag{4.7}$$

$$2F_{[uw]} = [(\Gamma-1)u+\Delta]A_{wu} + w(A_{ww} - A_{uu}) + vA_{vw} = 0 \tag{4.8}$$

$$2F_{[vw]} = vA_{uw} - wA_{uv} = 0 \tag{4.9}$$

where $F_{[uv]} = \frac{1}{2}(F_{uv} - F_{vu})$.

Subtracting $w$ times equation (4.7) from $v$ times equation (4.8), and using equation (4.9) leads to the equation

$$(v^2 - w^2)A_{vw} + vw(A_{ww} - A_{vv}) = 0. \tag{4.10}$$

Converting to polar coordinates, equations (4.9) and (4.10) may be written in the form

$$\frac{\partial A_u}{\partial\theta} = 0 \qquad \frac{\partial}{\partial\theta}\left(r\frac{\partial A}{\partial r} - A\right) = 0. \tag{4.11}$$

Equations (4.11) have solutions of the form

$$A = a(r,u) + rh(\theta). \tag{4.12}$$

Substituting the solution ansatz (4.12) in equation (4.8) requires that $a(r,u)$ satisfies the linear second-order partial differential equation

$$[(\Gamma-1)u+\Delta]a_{ru} + r(a_{rr} - a_{uu}) = 0. \tag{4.13}$$

Equation (4.13) plays a central role in the following analysis. It is of interest to note that the solution

$$A = rh(\theta) \tag{4.14}$$

obtained by setting $a(r, u) = 0$ in equation (4.12) corresponds to the conservation law

$$D_t[rh(\theta)] + D_x[(u - \Delta)rh(\theta)] = 0 \tag{4.15}$$

which may be verified by using equations (4.3).

Now consider in detail the solutions for $A$ of the form $A = a(r, u)$, where $a(r, u)$ satisfies the partial differential equation (4.13). In this case equations (4.4)–(4.6) for the fluxes satisfy the equations

$$F_u = \Gamma u A_u + r A_r \qquad F_r = r A_u + (u - \Delta) A_r \qquad F_\theta = 0. \tag{4.16}$$

The integrability conditions for equations (4.16) imply that $a(r, u)$ must satisfy equation (4.13). The characteristic coordinates for the hyperbolic equation (4.13) are functions $\xi = \xi(r, u)$ satisfying the characteristic equation

$$[(\Gamma - 1)u + \Delta]\xi_r \xi_u + r(\xi_r^2 - \xi_u^2) = 0. \tag{4.17}$$

Solving equation (4.17) for $\xi_r/\xi_u$ yields the two characteristic equations

$$\frac{\xi_r}{\xi_u} = -\frac{(\Gamma u - \lambda_\pm)}{r} \qquad \text{or} \qquad r\xi_r + (\Gamma u - \lambda_\pm)\xi_u = 0 \tag{4.18}$$

where $\lambda_\pm$ are the characteristic speeds for the fast and slow modes listed in equations (3.6). Equations (4.18) are readily recognized as the Riemann invariant equations (3.11). The characteristics of equations (4.18) are given by equations (3.14). Thus the characteristic variables for equation (4.13) are just the Riemann invariants for the fast and slow modes obtained in section 3.

The analysis of the Riemann invariants in section 3 suggests that for $\Gamma \neq 1$ it is useful to write equation (4.13) in terms of the variable

$$\mu = \frac{(\Gamma - 1)u + \Delta}{r} \tag{4.19}$$

and $r$ as independent variables, to obtain the equation

$$[(2 - \Gamma)\mu^2 - (\Gamma - 1)^2]a_{\mu\mu} + (\Gamma - 3)\mu r a_{r\mu} + r^2 a_{rr} + (3 - \Gamma)\mu a_\mu = 0. \tag{4.20}$$

Equation (4.20) admits separable solutions of the form

$$a = r^{-s} G(\mu, s) \tag{4.21}$$

where $G$ satisfies the second-order differential equation

$$[(2 - \Gamma)\mu^2 - (\Gamma - 1)^2]G_{\mu\mu} - (s + 1)(\Gamma - 3)\mu G_\mu + s(s + 1)G = 0. \tag{4.22}$$

For $\Gamma \neq 2$, equation (4.22) has solutions in terms of hypergeometric functions in the form [28, ch 15, p 556]

$$G = \alpha_1 F\left(\frac{s}{2}, \frac{s+1}{2(2-\Gamma)}; \frac{1}{2}; z\right) + \beta_1 z^{\frac{1}{2}} F\left(\frac{s+3-\Gamma}{2(2-\Gamma)}, \frac{s+1}{2}; \frac{3}{2}; z\right) \tag{4.23}$$

where

$$z = \frac{(2-\Gamma)\mu^2}{(\Gamma-1)^2}. \tag{4.24}$$

Since the first hypergemetric function $F(a, b; c; z)$ solution term in equation (4.23) has it's third argument $c = \frac{1}{2}$, then the solutions for $G$ may also be written in terms of the solutions of Legendre's equation in the form [28, formula 15.4.23, p 562]

$$G = (z-1)^{\frac{\sigma}{2}} \left(\alpha_1 P_\nu^\sigma(z^{\frac{1}{2}}) + \beta_1 Q_\nu^\sigma(z^{\frac{1}{2}})\right) \tag{4.25}$$

where

$$\sigma = \frac{s(\Gamma-3)-(\Gamma-1)}{2(2-\Gamma)} \qquad \nu = \frac{\Gamma-3-s(\Gamma-1)}{2(2-\Gamma)} \tag{4.26}$$

and $P_\nu^\sigma(x)$ and $Q_\nu^\sigma(x)$ are standard solutions of Legendre's equation [28, ch 8, p 331]

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - \frac{2x}{1-x^2}\frac{\mathrm{d}y}{\mathrm{d}x} + \left(\frac{\nu(\nu+1)}{1-x^2} - \frac{\sigma^2}{(1-x^2)^2}\right) y = 0\,. \tag{4.27}$$

It is of interest to note that the separated solutions (4.21) reduce to polynomial conserved densities for special values of the separation constant $s$. If one of the parameters $a$ or $b$ in the hypergeometric function $F(a, b; c; z)$ are negative integers, then the hypergeometric power series terminates leading to polynomial conserved densities. In particular for $s = -2n$ and $s = -2n - 1$ one obtains polynomial conserved densities

$$\begin{aligned} a_{2n} &= \alpha_1 \frac{n!}{(1/2)_n} r^{2n} P_n^{(-\frac{1}{2},\gamma_1)}(1-2z) \\ a_{2n+1} &= \beta_1 \frac{n!}{(3/2)_n} r^{2n+1} z^{\frac{1}{2}} P_n^{(\frac{1}{2},\gamma_2)}(1-2z) \end{aligned} \tag{4.28}$$

where

$$\begin{aligned} \gamma_1 &= \frac{2n-1}{2(\Gamma-2)} - \left(n + \frac{1}{2}\right) \\ \gamma_2 &= \frac{2n+\Gamma-2}{2(\Gamma-2)} - \left(n + \frac{3}{2}\right) \\ (q)_n &= q(q+1) + \cdots (q+n-1) \end{aligned} \tag{4.29}$$

and $P_n^{(\alpha,\beta)}(x)$ is the Jacobi polynomial of degree $n$ [28, formula 15.4.6, p 561]. The solutions (4.28) can also be written in terms of Gegenbauer or ultraspherical polynomials. The polynomial $a_{2n}(u, r)$ is of degree $2n$ in both $u$ and $r$, whereas $a_{2n+1}(u, r)$ is of degree $2n + 1$ in both $u$ and $r$. We show in the appendix how the conserved densities $A_1$ and $A_2$ associated with space translation and time translation invariance given in equations (2.21) and (2.22) (in the dispersionless case), namely

$$A_1 = \tfrac{1}{2}(u^2 + r^2) \qquad A_2 = \tfrac{1}{2}\left(\tfrac{1}{3}\Gamma u^3 + (u-\Delta)r^2\right) \tag{4.30}$$

are related to the polynomial densities (4.28).

The solutions (4.21)–(4.23) do not cover the special cases $\Gamma = 1$ and $\Gamma = 2$ which are dealt with below.

*The Γ = 2 case*

In this case equation (4.22) reduces to

$$G_{\mu\mu} - (s+1)\mu G_{\mu} - s(s+1)G = 0. \tag{4.31}$$

Equation (4.31) has solutions in terms of parabolic cylinder functions (which are a special case of the confluent hypergeometric function) in the form

$$G = \exp\left(\frac{(s+1)\mu^2}{4}\right)\left(\alpha_1 E^0_{-s}[(s+1)^{\frac{1}{2}}\mu] + \beta_1 D_{-s}[(s+1)^{\frac{1}{2}}\mu]\right) \tag{4.32}$$

where $E^0_{\nu}(x)$ and $D_{\nu}(x)$ are standard parabolic cylinder functions [28, ch 19, p 685]. For $s = -n$ one obtains the polynomial conserved densities

$$a(r,u) = \alpha_1 r^n He_n[(1-n)^{\frac{1}{2}}\mu] \tag{4.33}$$

where $He_n(x)$ is a standard Hermite polynomial [28, ch 22].

*The Γ = 1 case*

In this case equation (4.13) has separable solutions of the form

$$a = \exp(\alpha u)Y(r,\alpha) \tag{4.34}$$

where $Y(r,\alpha)$ satisfies the differential equation

$$\frac{\mathrm{d}^2Y}{\mathrm{d}r^2} + \frac{\alpha\Delta}{r}\frac{\mathrm{d}Y}{\mathrm{d}r} - \alpha^2 Y = 0. \tag{4.35}$$

Solving equation (4.35) for $Y$ leads to separated solutions for $a$ of the form

$$a = \exp[\alpha(u-r)]\left[a_1 M\left(\tfrac{1}{2}\alpha, \alpha\Delta, 2\alpha r\right) + b_1 U\left(\tfrac{1}{2}\alpha, \alpha\Delta, 2\alpha r\right)\right] \tag{4.36}$$

where $M(a,b,x)$ and $U(a,b,x)$ are two standard confluent hypergeometric functions [28, ch 13]. One can obtain polynomial conserved densities from the solution (4.36) by expanding the solutions in a power series in $\alpha$.

For the special case $\Delta = 1$ the solution (4.36) may be expressed in terms of modified Bessel functions

$$a = \exp(\alpha u)r^{-\nu}[a_1 I_{\nu}(\alpha r) + b_1 K_{\nu}(\alpha r)] \tag{4.37}$$

where $\nu = (\alpha - 1)/2$, and $I_{\nu}(x)$ and $K_{\nu}(x)$ are modified Bessel functions of the first and second kind. From expanding the solution (4.37) in a power series in $\alpha$ one can obtain not only polynomial conserved densities in $u$ and $r$, but also polynomials in $u$, $r$ and $\ln r$.

## 5. Symmetries and conserved densities

In this section we use the theory of symmetries of systems of hydrodynamic type developed by Tsarev [22, 23] and others (see, e.g., [24]) to relate the infinite families of conservation laws for the dispersionless TDNLS equations obtained in section 4 to the Lie–Bäcklund symmetries admitted by the equations.

The first step in the analysis is to determine the eigenvalues and Riemann invariants of the hydrodynamic system of interest, and write the equations in terms of the Riemann invariants $\{R^i\}$ as dependent variables. This leads to the diagonalized system of equations for the Riemann invariants

$$R^i_t + \lambda^i(R^\alpha)R^i_x = 0 \qquad i, \alpha = 1(1)N \tag{5.1}$$

where the eigenvalues $\lambda^i(R^\alpha)$ are implicit functions of the Riemann invariants. The eigenvalues and Riemann invariants for the dispersionless TDNLS equations have already been established in section 3.

The second step in the analysis is to investigate if the system (5.1) admits a commuting flow or symmetry, in which the Riemann invariants satisfy the associated diagonalized system

$$R^i_\tau + W^i(R^\alpha)R^i_x = 0 \qquad i = 1(1)N \tag{5.2}$$

where the $R^i(x, t, \tau)$ is regarded as a function of the new 'time' variable $\tau$, in which translations with respect to $\tau$ correspond to a Lie–Bäcklund symmetry. The system of equations (5.1) and (5.2) are compatible in this sense if

$$R^i_{t\tau} = R^i_{\tau t}. \tag{5.3}$$

Using equations (5.1) and (5.2) to eliminate derivatives with respect to $t$ and $\tau$ in equations (5.3) yields the system of first-order partial differential equations

$$\frac{\partial_j W^i}{W^j - W^i} = \frac{\partial_j \lambda^i}{\lambda^j - \lambda^i} \qquad i \neq j \quad i, j = 1(1)N \tag{5.4}$$

where we use the notation $\partial_j W^i = \partial W^i/\partial R^j$ and $\partial_j\lambda^i = \partial\lambda^i/\partial R^j$. The integrability conditions for the differential equation system (5.4): $W^i_{jk} = W^i_{kj}$ leads [22–24] to the equations

$$T^i_{jk} \equiv \partial_k\left(\frac{\partial_j\lambda^i}{\lambda^j - \lambda^i}\right) - \partial_j\left(\frac{\partial_k\lambda^i}{\lambda^k - \lambda^i}\right) = 0 \qquad i \neq j \neq k. \tag{5.5}$$

If the conditions (5.5) are satisfied, then the original system (5.1) admits an infinite number of Lie–Bäcklund symmetries, with canonical Lie–Bäcklund operators of the form

$$\hat{X} = \hat{S}^\alpha \frac{\partial}{\partial R^\alpha} + \zeta^\alpha_{\mathrm{s}} \frac{\partial}{\partial R^\alpha_{\mathrm{s}}} + \zeta^\alpha_{sp} \frac{\partial}{\partial R^\alpha_{sp}} + \cdots \tag{5.6}$$

where [29]

$$\hat{S}^\alpha = -R^\alpha_\tau = W^\alpha R^\alpha_x \qquad \zeta^\alpha_{\mathrm{s}} = D_{\mathrm{s}}(\hat{S}^\alpha) \qquad \zeta^\alpha_{sp} = D_{\mathrm{s}}D_{\mathrm{p}}(\hat{S}^\alpha) \quad \cdots \tag{5.7}$$

and the operators $\{D_s : s = 1, 2\}$ correspond to total derivatives with respect to $x$ and $t$ respectively. In more classical terminology, this implies that equations (5.1) are invariant under the canonical infinitesimal Lie transformations

$$R'^{\alpha} = R^{\alpha} + \epsilon \hat{S}^{\alpha} \qquad x' = x \qquad t' = t. \tag{5.8}$$

If the conditions (5.5) are satisfied the system admits an infinite number of commuting flows. A diagonal hydrodynamic system of the form (5.1), in which the $\lambda^i$ are distinct and non-zero, consisting of $N \geqslant 3$ equations, which is not necessarily Hamiltonian, and which satisifies conditions (5.4) is called semi-Hamiltonian. Such systems admit an infinite number of commuting flows. It is worth noting that the generalized hodograph transformation

$$W^i(R) = x - \lambda^i(R)t \tag{5.9}$$

for $W^i(R)$ a solution of equations (5.4) may be used to obtain the general solution of equations (5.2) [23, 24].

In the next section we show that the dispersionless TDNLS system (3.1) is a semi-Hamiltonian system by verifying that the conditions (5.5) hold for the equations. In section 5.2, we solve the differential equation system (5.4) for the $W^i$. The corresponding Lie–Bäcklund symmetries are then determined from equations (5.6)–(5.8). Using the general theory of Hamiltonian systems (e.g. [30, 25]) the symmetries are then related to the infinite class of conserved densities derived in section 4.

### *5.1. The integrability conditions (5.5) and derivative transformations*

Taking $\lambda^1 < \lambda^2 < \lambda^3$ to correspond to the slow, intermediate and fast mode waves (see section 3), we find

$$\frac{\partial}{\partial R^1} = \frac{\partial\mu}{\partial R^1}\left(\frac{\partial}{\partial\mu} + \frac{2r}{(\Gamma-3)\mu - (\Gamma-1)(\mu^2+4)^{\frac{1}{2}}}\frac{\partial}{\partial r}\right) \tag{5.10}$$

$$\frac{\partial}{\partial R^3} = \frac{\partial\mu}{\partial R^3}\left(\frac{\partial}{\partial\mu} + \frac{2r}{(\Gamma-3)\mu + (\Gamma-1)(\mu^2+4)^{\frac{1}{2}}}\frac{\partial}{\partial r}\right) \tag{5.11}$$

$$\frac{\partial}{\partial R^2} = \cos^2\theta\frac{\partial}{\partial\theta} \tag{5.12}$$

for the partial derivative operators $\{\partial/\partial R^i : i = 1(1)3\}$, where

$$\frac{\partial\mu}{\partial R^1} = -\frac{(\Gamma-1)[1+(\Gamma-2)\mu^2/(\Gamma-1)^2]}{R^1(\mu^2+4)^{\frac{1}{2}}} \tag{5.13}$$

$$\frac{\partial\mu}{\partial R^3} = \frac{(\Gamma-1)[1+(\Gamma-2)\mu^2/(\Gamma-1)^2]}{R^3(\mu^2+4)^{\frac{1}{2}}}. \tag{5.14}$$

In equations (5.10)–(5.14), $R^1$, $R^2$, and $R^3$ refer to the Riemann invariants for the slow, intermediate, and fast modes, and the variable $\mu$ is defined in terms of $r$ and $u$ in equation (3.16). The differential operator $\partial/\partial R^1$ corresponds to differentiation along the fast mode characteristic $R^3 =$ constant, whereas $\partial/\partial R^3$ corresponds to differentiation along the slow mode characteristic $R^1 =$ constant. Using $r$ and $u$ as independent variables in equations (5.10) and (5.11) instead of $r$ and $\mu$ yields the formulae

$$\frac{\partial}{\partial R^1} = \frac{2r(\partial\mu/\partial R^1)}{(\Gamma-3)\mu-(\Gamma-1)(\mu^2+4)^{\frac{1}{2}}}\left(\frac{\partial}{\partial r}+\frac{\lambda^1-\lambda^2}{r}\frac{\partial}{\partial u}\right) \tag{5.15}$$

$$\frac{\partial}{\partial R^3} = \frac{2r(\partial\mu/\partial R^3)}{(\Gamma-3)\mu+(\Gamma-1)(\mu^2+4)^{\frac{1}{2}}}\left(\frac{\partial}{\partial r}+\frac{\lambda^3-\lambda^2}{r}\frac{\partial}{\partial u}\right). \tag{5.16}$$

We also note for later reference the formulae

$$\lambda^2-\lambda^1 = \tfrac{1}{2}r[(\mu^2+4)^{\frac{1}{2}}-\mu] \qquad \lambda^3-\lambda^2 = \tfrac{1}{2}r[(\mu^2+4)^{\frac{1}{2}}+\mu] \qquad \lambda^3-\lambda^1 = r(\mu^2+4)^{\frac{1}{2}} \tag{5.17}$$

for the eigenvalue differences.

Using the results (3.6) for the characteristic wave speeds $\lambda^1$, $\lambda^2$ and $\lambda^3$, and the results (5.10)–(5.12) we obtain the formulae

$$\begin{aligned}
\frac{\partial\lambda^2}{\partial R^1} &= -\frac{(\Gamma-1)r[1+(\Gamma-2)\mu^2/(\Gamma-1)^2][\mu-(\mu^2+4)^{\frac{1}{2}}]}{R^1(\mu^2+4)^{\frac{1}{2}}[(\Gamma-3)\mu-(\Gamma-1)(\mu^2+4)^{\frac{1}{2}}]}\\
\frac{\partial\lambda^3}{\partial R^1} &= \frac{(\Gamma-1)r[1+(\Gamma-2)\mu^2/(\Gamma-1)^2][2(\Gamma-1)+\mu^2-\mu(\mu^2+4)^{\frac{1}{2}}]}{R^1(\mu^2+4)[(\Gamma-3)\mu-(\Gamma-1)(\mu^2+4)^{\frac{1}{2}}]}\\
\frac{\partial\lambda^1}{\partial R^2} &= \frac{\partial\lambda^2}{\partial R^2} = \frac{\partial\lambda^3}{\partial R^2} = 0\\
\frac{\partial\lambda^1}{\partial R^3} &= \frac{(\Gamma-1)r[1+(\Gamma-2)\mu^2/(\Gamma-1)^2][2(\Gamma-1)+\mu^2+\mu(\mu^2+4)^{\frac{1}{2}}]}{R^3(\mu^2+4)[(\Gamma-3)\mu+(\Gamma-1)(\mu^2+4)^{\frac{1}{2}}]}\\
\frac{\partial\lambda^2}{\partial R^3} &= \frac{(\Gamma-1)r[1+(\Gamma-2)\mu^2/(\Gamma-1)^2][\mu+(\mu^2+4)^{\frac{1}{2}}]}{R^3(\mu^2+4)^{\frac{1}{2}}[(\Gamma-3)\mu+(\Gamma-1)(\mu^2+4)^{\frac{1}{2}}]}
\end{aligned} \tag{5.18}$$

for the $\lambda^j$ derivatives appearing in equations (5.4).

Using the results (5.17), (5.18), it is straightforward to verify that equations (5.5) are satisfied. Hence the dispersionless TDNLS system is a semi-Hamiltonian system.

### *5.2. Lie–Bäcklund symmetries*

Using the results (5.18) for the $\partial\lambda^i/\partial R^j$ and the partial derivative formulae (5.12), (5.15) and (5.16) for the derivatives with respect to the Riemann invariants, the differential equation system (5.4) for the $W^i$ reduces to

$$r\frac{\partial W^2}{\partial r}+(\lambda^1-\lambda^2)\frac{\partial W^2}{\partial u} = W^1-W^2 \tag{5.19}$$

$$r\frac{\partial W^3}{\partial r}+(\lambda^1-\lambda^2)\frac{\partial W^3}{\partial u} = (W^1-W^3)\frac{[2(\Gamma-1)+\mu^2-\mu(\mu^2+4)^{\frac{1}{2}}]}{2(\mu^2+4)} \tag{5.20}$$

$$\frac{\partial W^1}{\partial\theta} = \frac{\partial W^3}{\partial\theta} = 0 \tag{5.21}$$

$$r\frac{\partial W^1}{\partial r}+(\lambda^3-\lambda^2)\frac{\partial W^1}{\partial u} = (W^3-W^1)\frac{[2(\Gamma-1)+\mu^2+\mu(\mu^2+4)^{\frac{1}{2}}]}{2(\mu^2+4)} \tag{5.22}$$

$$r\frac{\partial W^2}{\partial r}+(\lambda^3-\lambda^2)\frac{\partial W^2}{\partial u} = W^3-W^2. \tag{5.23}$$

Equations (5.19) and (5.23) may be re-arranged to yield expressions for $W^1$ and $W^3$ in terms of $W^2$

$$W^1 = W^2 + r\frac{\partial W^2}{\partial r} + (\lambda^1 - \lambda^2)\frac{\partial W^2}{\partial u} \tag{5.24}$$

$$W^3 = W^2 + r\frac{\partial W^2}{\partial r} + (\lambda^3 - \lambda^2)\frac{\partial W^2}{\partial u}. \tag{5.25}$$

Using the results (5.24) and (5.25) for $W^1$ and $W^3$ in equation (5.22) yields a linear, second-order hyperbolic equation for $W^2$

$$[(\Gamma - 1)u + \Delta]W^2_{ur} + r(W^2_{rr} - W^2_{uu}) + 2W^2_r = 0. \tag{5.26}$$

Exactly the same equation for $W^2$ is obtained if $W^1$ and $W^3$ are eliminated from equation (5.20).

It is of interest to note that equation (5.26) is similar to the conserved density equation (4.13) for $a(r,u)$ (the two equations have the same higher-order derivative terms, but equation (5.26) has a first-order derivative term not present in equation (4.13)). This suggests that there is a direct link between the solutions for $W^2$ and the conserved densities $A$ obtained in section 4.

Equations (5.24)–(5.26) imply that the general solution for $W^2$ is of the form

$$W^2 = \frac{p(\theta)}{r} + \hat{W}^2(u,r) \tag{5.27}$$

where $\hat{W}^2(u,r)$ satisfies equation (5.26).

One can obtain separated solutions of equation (5.26) by converting to $r$ and $\mu$ as independent variables, to obtain the equation

$$[(2-\Gamma)\mu^2 - (\Gamma-1)^2]W^2_{\mu\mu} + (\Gamma-3)r\mu W^2_{r\mu} + r^2W^2_{rr} - (\Gamma-1)\mu W^2_\mu + 2rW^2_r = 0. \tag{5.28}$$

Equation (5.28) has separated solutions of the form

$$W^2 = r^{-s}Y(\mu) \tag{5.29}$$

where $Y(\mu)$ satisfies the ordinary differential equation

$$[(2-\Gamma)\mu^2 - (\Gamma-1)^2]\frac{\mathrm{d}^2Y}{\mathrm{d}\mu^2} - [s(\Gamma-3) + \Gamma - 1]\mu\frac{\mathrm{d}Y}{\mathrm{d}\mu} + s(s-1)Y = 0. \tag{5.30}$$

Equation (5.30) has solutions in terms of hypergeometric functions

$$Y = a_1F\left(\frac{s}{2}, \frac{1-s}{2(\Gamma-2)}; \frac{1}{2}; z\right) + a_2z^{\frac{1}{2}}F\left(\frac{s+1}{2}, \frac{\Gamma-1-s}{2(\Gamma-2)}; \frac{3}{2}; z\right) \tag{5.31}$$

where

$$z = \frac{(2-\Gamma)\mu^2}{(\Gamma-1)^2}. \tag{5.32}$$

The solution (5.31) applies for $\Gamma \neq 1$ and $\Gamma \neq 2$. The solution (5.31) can also be expressed in terms of Legendre functions.

In the analysis below we show explicitly the link between the above solutions for $W^2$ and the conserved densities obtained in section 4.

*5.3. The link between symmetries and conserved densities*

By reverting to $(u, v, w)$ as dependent variables, the Lie–Bäcklund operator (5.6) takes the form

$$\hat{X} = \sum_{\alpha=1}^{3} \hat{S}^\alpha \frac{\partial}{\partial u^\alpha} + \cdots \tag{5.33}$$

where

$$\hat{S}^\alpha = \sum_{s=1}^{3} \frac{\partial u^\alpha}{\partial R^s} W^s R_x^s. \tag{5.34}$$

Since the TDNLS system is a Hamiltonian system with canonical variables $(u, v, w)$ and symplectic operator $D_x$ (see equations (2.25) *et seq.*), the symmetries $\hat{S}^\alpha$ are related to the potential operators by [25] the equations

$$\hat{S}^\alpha = D_x(Q^\alpha)\,. \tag{5.35}$$

The potential operators $Q^\alpha$ in turn are related to the conserved densities $A$ and functionals $\mathcal{A}$ by the equations

$$Q^\alpha = \frac{\delta \mathcal{A}}{\delta u^\alpha} \qquad \mathcal{A} = \int_{-\infty}^{\infty} A\,\mathrm{d}x. \tag{5.36}$$

The conserved densities $A$ can be determined by noting that

$$F_Q[u] = \int_0^1 \mathrm{d}s \left\langle \mathbf{Q}, \frac{\partial \boldsymbol{u}}{\partial s} \right\rangle = \int_{-\infty}^{\infty} [A(u)]_{s=0}^{s=1}\,\mathrm{d}x \tag{5.37}$$

where the inner product [30, 25]

$$\left\langle \mathbf{Q}(u), \frac{\partial \boldsymbol{u}}{\partial s} \right\rangle = \int_{-\infty}^{\infty} \sum_{\alpha=1}^{3} Q^\alpha \frac{\partial u^\alpha}{\partial s}\,\mathrm{d}x\,. \tag{5.38}$$

In equation (5.37), $\delta u^\alpha = (\partial u^\alpha/\partial s)\mathrm{d}s$ corresponds to the variation of $u^\alpha$ in the calculus of variations sense. Our aim in the present section is to first determine in detail the form of the relation (5.35) between the symmetries $\hat{S}^\alpha$ and potential operators $Q^\alpha$. Once the $Q^\alpha$ are established, the conserved densities are determined from a consideration of the functionals $F_Q[u]$ defined in equations (5.37). The Poisson bracket for functionals

$$\{F_P[u], F_Q[u]\} = \int_{-\infty}^{\infty} \frac{\delta F_P}{\delta u^\alpha} D_x\left(\frac{\delta F_Q}{\delta u^\alpha}\right)\mathrm{d}x = \int_{-\infty}^{\infty} P^\alpha D_x(Q^\alpha)\,\mathrm{d}x \tag{5.39}$$

may then be used to study the Poisson bracket Lie algebra.

*(a) The $W^2 = W^2(u,r)$ case.* Using the results (5.24) and (5.25) for $W^1$ and $W^3$ in equation (5.34) leads to the formulae

$$\hat{S}^\alpha = W^2\left(\frac{\partial u^\alpha}{\partial R^2}R^2_x + \frac{\partial u^\alpha}{\partial R^1}R^1_x + \frac{\partial u^\alpha}{\partial R^3}R^3_x\right) + rW^2_r\left(\frac{\partial u^\alpha}{\partial R^1}R^1_x + \frac{\partial u^\alpha}{\partial R^3}R^3_x\right)$$
$$+ W^2_u\left[(\lambda^1-\lambda^2)\frac{\partial u^\alpha}{\partial R^1}R^1_x + (\lambda^3-\lambda^2)\frac{\partial u^\alpha}{\partial R^3}R^3_x\right] \qquad \alpha = 1(1)3. \tag{5.40}$$

Use of the fact that the Riemann invariants satisfy the diagonalized system (5.1), and using the dispersionless TDNLS equations (4.2) to eliminate the time derivatives $u_t$, $v_t$ and $w_t$, equations (5.40) reduce to

$$\hat{S}^1 = \{W^2 + rW^2_r + [(\Gamma-1)u + \Delta]W^2_u\}u_x + rW^2_u r_x \tag{5.41}$$
$$\hat{S}^2 = D_x[W^2 v] \tag{5.42}$$
$$\hat{S}^3 = D_x[W^2 w]. \tag{5.43}$$

Equations (5.41)–(5.43) only apply for the case where $W^2 = W^2(u,r)$, and modified versions of these formulae apply for the case where $W^2 = p(\theta)/r$.

Further consideration of equation (5.41) shows that we may write $\hat{S}^1 = D_x(Q^1)$ for an appropriate potential $Q^1$, where

$$\frac{\partial Q^1}{\partial u} = W^2 + rW^2_r + [(\Gamma-1)u + \Delta]W^2_u \tag{5.44}$$
$$\frac{\partial Q^1}{\partial r} = rW^2_u. \tag{5.45}$$

The integrability condition $Q^1_{ur} = Q^1_{ru}$ in these equations requires that $W^2$ satisfy equation (5.26), which is automatically satisfied. Hence from equations (5.41)–(5.45) we obtain

$$Q^1 = \int_{r_0}^{r} r' W^2_u(u,r')\,\mathrm{d}r' + \int_{u_0}^{u}[W^2(u',r) + rW^2_r(u',r)$$
$$+ [(\Gamma-1)u' + \Delta]W^2_{u'}(u',r)]_{r_0}\,\mathrm{d}u' \tag{5.46}$$
$$Q^2 = vW^2 \qquad Q^3 = wW^2$$

for the potential operators $Q^\alpha$ in equations (5.35). The corresponding symmetry operator is

$$\hat{X}_{W^2(u,r)} = D_x(Q^1)\frac{\partial}{\partial u} + D_x[vW^2(u,r)]\frac{\partial}{\partial v} + D_x[wW^2(u,r)]\frac{\partial}{\partial w} + \cdots \tag{5.47}$$

where $W^2(u,r)$ satisfies equation (5.26).

From equations (5.37), (5.38), the functional $F_Q[u]$ has the form

$$F_Q[u] = \int_0^1 \mathrm{d}s\int_{-\infty}^{\infty}\mathrm{d}x\left(Q^1\frac{\partial u}{\partial s} + Q^2\frac{\partial v}{\partial s} + Q^3\frac{\partial w}{\partial s}\right). \tag{5.48}$$

The functional (5.48) may be re-written as

$$F_Q[u] = \int_0^1 \mathrm{d}s \int_{-\infty}^{\infty} \mathrm{d}x \left[ \frac{\partial}{\partial s}(uQ^1) + \frac{\partial r}{\partial s} r(W^2 - uW_u^2) - \frac{\partial u}{\partial s} u(W^2 + rW_r^2 + [(\Gamma - 1)u + \Delta]W_u^2) \right]. \tag{5.49}$$

The terms involving $\partial r/\partial s$ and $\partial u/\partial s$ in equation (5.49) may be written as the gradient of a potential $P(u, r)$ where

$$P(u, r) = \int_{r_0}^{r} r'[W^2(u, r') - uW_u^2(u, r')]\,\mathrm{d}r' - \int_{u_0}^{u} \mathrm{d}u' \Big[ W^2(u', r) + rW_r^2(u', r) + [(\Gamma - 1)u' + \Delta]W_{u'}^2(u', r) \Big]_{r=r_0}. \tag{5.50}$$

Taking into account the result (5.50), equation (5.49) reduces to

$$F_Q[u] = \int_{-\infty}^{\infty} [uQ^1 + P]_{s=0}^{s=1}\,\mathrm{d}x = \int_{-\infty}^{\infty} [A]_{s=0}^{s=1}\,\mathrm{d}x. \tag{5.51}$$

From equation (5.51) we identify $A = uQ^1 + P$. Equations (5.46) and (5.50) then yield

$$A = \int_{r_0}^{r} r'W^2(u, r')\,\mathrm{d}r' + \int_{u_0}^{u} \mathrm{d}u'\,(u - u')[W^2(u', r) + rW_r^2(u', r) + [(\Gamma - 1)u' + \Delta]W_{u'}^2(u', r)]_{r_0} \tag{5.52}$$

as the conserved density corresponding to the symmetry operator (5.33). It is straightforward to verify that the conserved density (5.52) satisfies the linear second-order partial differential equation (4.13) for the conserved density $A(u, r)$. Thus we have obtained a direct link between the symmetries associated with the $\{W^\alpha : \alpha = 1(1)3\}$ and the corresponding conserved density $A$.

*(b) The $W^2 = p(\theta)/r$ case*. Starting from equations (5.24), (5.25) the above solution ansatz implies

$$W^1 = W^3 = 0. \tag{5.53}$$

Equations (5.34) then reduce to

$$\hat{S}^\alpha = \frac{\partial u^\alpha}{\partial R^2} W^2 R_x^2 \qquad \alpha = 1(1)3. \tag{5.54}$$

From equations (5.54) we obtain

$$\hat{S}^\alpha = D_x(Q^\alpha) \qquad \alpha = 1(1)3 \tag{5.55}$$

where

$$Q^1 = 0 \qquad Q^2 = -\int_{\theta_0}^{\theta} p(\theta')\sin\theta'\,\mathrm{d}\theta' \qquad Q^3 = \int_{\theta_0}^{\theta} p(\theta')\cos\theta'\,\mathrm{d}\theta' \tag{5.56}$$

are the potential operators.

A calculation similar to that carried out in case (a), leads to the identification of the conserved density

$$A = rh(\theta) \tag{5.57}$$

where

$$h(\theta) = \int_{\theta_0}^{\theta} p(\theta') \sin(\theta - \theta')\, \mathrm{d}\theta'. \tag{5.58}$$

The conserved density (5.57) corresponds to the conserved density (4.14) obtained by more elementary means in section 4. The present approach shows explicitly the connection between the conserved density and the corresponding symmetry. The Lie–Bäcklund symmetry operator (5.33) in cylindrical polar coordinates may be written in the form

$$\hat{X}_{p(\theta)} = D_x(Q^2)\frac{\partial}{\partial v} + D_x(Q^3)\frac{\partial}{\partial w} \equiv \frac{p(\theta)\theta_x}{r}\frac{\partial}{\partial \theta} + \cdots \tag{5.59}$$

where $Q^2$ and $Q^3$ are given by equations (5.56). The symmetry (5.59) is a non-local symmetry.

## 6. Lie–Bäcklund symmetries and the hodograph transformation

The aim of this section is to show how the function $W^2(u, r)$ associated with the Lie–Bäcklund symmetries of equations (5.6) and (5.24)–(5.26) arises from a consideration of the hodograph transformation of the dispersionless TDNLS equations. It is also shown how the hodograph transformation leads to the symmetries (5.41)–(5.43) and conserved densities (5.52) obtained in section 5.

We note that the point Lie symmetries admitted by the dispersionless TDNLS system (4.3) for $\Gamma \neq 1$ has the general isovector

$$X = c_1X_1 + c_2X_2 + h(\theta)X_3 + c_4X_4 + c_5X_5 \tag{6.1}$$

where the symmetry operators $X_1$, $X_2$, $X_3$ and $X_4$ are given by equations (2.15) (the Lie point symmetry operators for the dispersive TDNLS equations), plus the 'stretch' operator

$$X_5 = x\frac{\partial}{\partial x} + t\frac{\partial}{\partial t}. \tag{6.2}$$

The general isovector of the dispersive TDNLS system is obtained by setting $h(\theta) = c_3 =$ constant, and $c_5 = 0$ in equation (6.1). The symmetry operator (6.1) also applies for the case $\Gamma = 1$ and $\Delta \neq 0$. For $\Gamma = 1$ and $\Delta = 0$, a larger symmetry algebra is obtained for the TDNLS system [20].

*6.1. A simplified version of the dispersionless TDNLS system*

In this section we consider the version of the dispersionless TDNLS system (4.3) that results when the solution for $\theta$ is taken as $\theta = \text{constant}$. Equations (4.3) then reduce to

$$\begin{aligned} &u_t + \Gamma u u_x + r r_x = 0 \\ &r_t + (u - \Delta) r_x + r u_x = 0. \end{aligned} \tag{6.3}$$

Equations (6.3) may be linearized by the hodograph transformation in which $t$ and $x$ are regarded as functions of $u$ and $r$. One method of obtaining the hodograph equations is to note that equations (6.3) may be represented by the closed ideal of differential forms (see, e.g., [31])

$$\begin{aligned} &\alpha_1 = \mathrm{d}x \wedge \mathrm{d}u + (\Gamma u \mathrm{d}u + r \mathrm{d}r) \wedge \mathrm{d}t \equiv \mathrm{d}[-u\mathrm{d}x + \tfrac{1}{2}(\Gamma u^2 + r^2)\mathrm{d}t] \\ &\alpha_2 = \mathrm{d}x \wedge \mathrm{d}r + [(u - \Delta)\mathrm{d}r + r\mathrm{d}u] \wedge \mathrm{d}t \equiv \mathrm{d}[-r\mathrm{d}x + r(u - \Delta)\mathrm{d}t]. \end{aligned} \tag{6.4}$$

Sectioning the forms (6.4) by setting $\alpha_1 = 0$ and $\alpha_2 = 0$, and requiring that $u$ and $r$ to be functions of $x$ and $t$ yields equations (6.3).

On the other hand sectioning the forms (6.4) by setting $\alpha_1 = 0$ and $\alpha_2 = 0$ and requiring $t = t(u, r)$ and $x = x(u, r)$ yields the hodograph equations

$$x_r = \Gamma u t_r - r t_u \qquad x_u = (u - \Delta) t_u - r t_r. \tag{6.5}$$

The integrability condition $x_{ru} = x_{ur}$ for equations (6.5) leads to the equation

$$r(t_{rr} - t_{uu}) + [(\Gamma - 1)u + \Delta] t_{ur} + (\Gamma + 1) t_r = 0 \tag{6.6}$$

for $t(u, r)$. The hodograph transformation remains single valued provided the Jacobian

$$J = \frac{\partial(x, t)}{\partial(u, r)} = x_u t_r - x_r t_u = r(t_u^2 - t_r^2) - [(\Gamma - 1)u + \Delta] t_r t_u \tag{6.7}$$

is non-zero. The Jacobian vanishes if $t = t(u, r)$ is a Riemann invariant of the dispersionless TDNLS system (i.e. if $t$ is constant on the characteristic curves (3.14)). The integrability condition $t_{ur} = t_{ru}$ yields the equation

$$\begin{aligned} &[\Gamma u(u - \Delta) - r^2]\{r(x_{rr} - x_{uu}) + [(\Gamma - 1)u + \Delta] x_{ur}\} + [2r^2 + \Gamma(2u - \Delta)(u - \Delta)] x_r \\ &\qquad + \Gamma r(4u - \Delta) x_u = 0 \end{aligned} \tag{6.8}$$

for $x = x(u, r)$. Equations (6.6) and (6.8) are linear second-order partial differential equations for $t(u, r)$ and $x(u, r)$. Hence the hodograph transformation provides a linearization of the simplified TDNLS system (6.3).

The form of equation (6.6) for $t(u, r)$ is reminiscent of equation (5.26) for the function $W^2(u, r)$ that arose in the analysis of the Lie–Bäcklund symmetries in section 5 (the two equations differ only in the first-order derivative terms with respect to $r$). To make the connection between $t(u, r)$ and $W^2(u, r)$ more concrete, note that the hodograph equations (6.5) may be formally integrated by setting $t = \Phi_u$ to yield the equations

$$t = \Phi_u \qquad x = (u - \Delta)\Phi_u - (\Phi + r\Phi_r) + G(r) \tag{6.9}$$

where $\Phi(u, r)$ satisfies the differential equation

$$r(\Phi_{rr} - \Phi_{uu}) + [(\Gamma - 1)u + \Delta]\Phi_{ur} + 2\Phi_r = 0 \tag{6.10}$$

and $G(r)$ is an arbitrary function of $r$. The equation (6.10) for $\Phi(u, r)$ is the same as equation (5.26) for $W^2(u, r)$. Thus the function $W^2(u, r)$ may be identified with $\Phi(u, r)$, and

$$t = \Phi_u \equiv W^2_u. \tag{6.11}$$

Equation (6.11) provides a link between the symmetry group analysis of section 5 and the hodograph transformation. Without loss of generality we set $G(r) = 0$ in equations (6.9) in the following analysis.

The above analysis indicates that the function $W^2 \equiv \Phi(u, r)$ is central to the analysis of the Lie–Bäcklund symmetries of the dispersionless TDNLS system (4.3). The Lie point symmetries of equation (6.10) for $\Phi(u, r)$, induce via hodograph transformation (6.9), symmetries of the dispersionless TDNLS equations (6.3). This situation is similar to the derivation of Lie–Bäcklund symmetries of the Burgers equation by exploiting the Cole–Hopf transformation between the Burgers equation and the heat equation [29], in which symmetries of the heat equation may be used to obtain symmetries of the Burgers equation.

Equation (6.10) admits the symmetry operator

$$Y = V^u \frac{\partial}{\partial u} + V^r \frac{\partial}{\partial r} + V^\Phi \frac{\partial}{\partial \Phi} \tag{6.12}$$

where

$$\begin{aligned} &V^u = c_4[(\Gamma - 1)u + \Delta] \qquad V^r = c_4(\Gamma - 1)r \\ &V^\Phi = [c_5 - c_4(\Gamma - 1)]\Phi + \Omega(u, r) \end{aligned} \tag{6.13}$$

and $\Omega(u, r)$ satisfies equation (6.10). The symmetry operator (6.12) may be written in the form

$$V = c_4 Y_4 + c_5 Y_5 + Y_\Omega \tag{6.14}$$

where

$$\begin{aligned} &Y_4 = (\Gamma - 1)r\frac{\partial}{\partial r} + [(\Gamma - 1)u + \Delta]\frac{\partial}{\partial u} - (\Gamma - 1)\Phi\frac{\partial}{\partial \Phi} \\ &Y_5 = \Phi\frac{\partial}{\partial \Phi} \qquad Y_\Omega = \Omega(u, r)\frac{\partial}{\partial \Phi}. \end{aligned} \tag{6.15}$$

Since $\Omega(u, r)$ is an arbitrary solution of equation (6.10), the symmetry algebra corresponding to the operators (6.15) is infinite dimensional.

The infinitesimal Lie transformations

$$t' = t + \epsilon V^t \qquad x' = x + \epsilon V^x \tag{6.16}$$

for $t$ and $x$ may be obtained from the hodograph transformation equations (6.9), and the Lie derivative extension formulae (e.g. [29])

$$V^{\Phi_{u^\beta}} = \frac{DV^\Phi}{Du^\beta} - \frac{DV^{u^\alpha}}{Du^\beta}\frac{\partial \Phi}{\partial u^\alpha} \tag{6.17}$$

where $(u^1, u^2) \equiv (u, r)$ and $D/Du^\beta$ denotes the total derivative with respect to $u^\beta$. The formulae for $V^t$ and $V^x$ are

$$\begin{aligned} V^t &= [c_5 - 2c_4(\Gamma - 1)]t + \Omega_u \\ V^x &= [c_5 - c_4(\Gamma - 1)]x + c_4\Gamma\Delta t + [(u - \Delta)\Omega_u - \Omega - r\Omega_r]\,. \end{aligned} \tag{6.18}$$

The parameter $c_4$ and the operator $Y_4$ may be identified with the symmetry operator $X_4$ in equations (2.15) for the dispersive TDNLS system. Similarly, $c_5$ and $Y_5$ may be identified with the 'stretch' symmetry $X_5$ in equation (6.2).

The infinite class of symmetries associated with $Y_\Omega$ in equation (6.15) where $\Omega(u, r)$ satisfies the partial differential equation (6.10) correspond in fact to the infinite class of Lie–Bäcklund symmetries (5.41)–(5.43) associated with the dispersionless TDNLS system (4.3). In particular we may identify the space and time translation symmetries $X_1 = \partial/\partial x$ and $X_2 = \partial/\partial t$ by noting that

$$\Omega = c_2(u - \Delta) - c_1 \tag{6.19}$$

is a solution of equation (6.10). Using equations (6.18) it follows that $\Omega = -1$ ($c_1 = 0$, $c_2 = 0$) corresponds to $X_1$ and $\Omega = u - \Delta$ ($c_1 = 0$, $c_2 = 1$) corresponds to $X_2$. Hence from equation (6.15) we find the operators

$$Y_1 = -\frac{\partial}{\partial\Phi} \qquad Y_2 = (u - \Delta)\frac{\partial}{\partial\Phi} \tag{6.20}$$

correspond to the space and time translation symmetries respectively of equations (6.3). The infinite dimensional symmetry algebra associated with $\{Y_1, Y_2, Y_4, Y_5, Y_\Omega\}$ and equation (6.10) has non-zero commutators

$$\begin{aligned} &[Y_1, Y_4] = -(\Gamma - 1)Y_1 \qquad [Y_1, Y_5] = Y_1 \\ &[Y_2, Y_4] = \Gamma\Delta Y_1 - 2(\Gamma - 1)Y_2 \qquad [Y_2, Y_5] = Y_2 \\ &[Y_4, Y_\Omega] = \Lambda\frac{\partial}{\partial\Phi} \equiv Y_\Lambda \qquad [Y_5, Y_\Omega] = -Y_\Omega \end{aligned} \tag{6.21}$$

where

$$\Lambda(u, r) = \mathcal{R}\Omega \qquad \mathcal{R} = [(\Gamma - 1)u + \Delta]\frac{\partial}{\partial u} + (\Gamma - 1)\left(r\frac{\partial}{\partial r} + 1\right). \tag{6.22}$$

One can verify that $\Lambda(u, r)$ satisfies the partial differential equation (6.10) for $\Phi(u, r)$, and hence $Y_\Lambda$ is an element of the Lie algebra. The commutators $[Y_1, Y_4]$ and $[Y_2, Y_4]$ in equations (6.21) obey the same commutation relations as $[X_1, X_4]$ and $[X_2, X_4]$ in the dispersive TDNLS Lie algebra (2.16). These results are expected in view of the fact that the two algebras are linked via the hodograph transformation (6.9).

*Proposition*. The operator $\mathcal{R}$ in equation (6.22) is a recursion operator for symmetries of the equation

$$K(\Phi) = \left[r\left(\frac{\partial^2}{\partial r^2} - \frac{\partial^2}{\partial u^2}\right) + [(\Gamma - 1)u + \Delta]\frac{\partial^2}{\partial u\partial r} + 2\frac{\partial}{\partial r}\right]\Phi = 0. \tag{6.23}$$

To prove the proposition, it is first necessary to define what is meant by a symmetry and a recursion operator.

*Definition 1.* The canonical Lie–Bäcklund symmetry operator $\hat{Y}(\hat{\eta})$ associated with the infinitesimal Lie transformation

$$x'^i = x^i + \epsilon\xi^i \qquad u'^\alpha = u^\alpha + \epsilon\eta^\alpha \qquad i = 1(1)n \quad \alpha = 1(1)m \tag{6.24}$$

is given by

$$\hat{Y}(\hat{\eta}) = \hat{\eta}^\alpha \frac{\partial}{\partial u^\alpha} + D_i(\hat{\eta}^\alpha)\frac{\partial}{\partial u_i^\alpha} + D_i D_j(\hat{\eta}^\alpha)\frac{\partial}{\partial u_{ij}^\alpha} + \cdots \tag{6.25}$$

where

$$\hat{\eta}^\alpha = \eta^\alpha - \xi^j u_j^\alpha \tag{6.26}$$

is the canonical Lie–Bäcklund symmetry and $D_i = D/Dx^i$ is the total derivative operator with respect to the independent variable $x^i$, and $u_i^\alpha = D_i(u^\alpha)$, $u_{ij}^\alpha = D_i D_j u^\alpha$, etc, denote the partial derivatives of the dependent variables $u^\alpha$ (see, e.g., [29]).

*Definition 2.* A symmetry $\hat{\eta}^\alpha$ of a differential equation system $K(u) = 0$ is an infinitesimal canonical Lie–Bäcklund transformation $x'^i = x^i$, $u'^\alpha = u^\alpha + \epsilon\hat{\eta}^\alpha$, which leaves the equation system invariant.

One can show (see, e.g., [29, 32]) that if $\hat{\eta}^\alpha$ is a symmetry of $K(u) = 0$, then

$$\hat{Y}(\hat{\eta})[K(u)]_{K=0} = 0 \tag{6.27}$$

where the subscript $K = 0$ in equation (6.27) emphasizes that $K(u) = 0$ on the solution manifold. Alternatively one can write

$$\hat{Y}(\hat{\eta})[K(u)] = K'(u)[\hat{\eta}] \tag{6.28}$$

where $K'(u)$ is the Fréchet derivative of $K(u)$.

*Definition 3.* An operator $\mathcal{R}$ is a recursion operator for symmetries of a differential equation system $K(u) = 0$, if $\mathcal{R}(\hat{\eta})$ is a symmetry whenever $\hat{\eta}$ is a symmetry.

Using the above definitions one can show that $\mathcal{R}$ is a recursion operator for $K(u) = 0$ iff

$$[\hat{Y}, \mathcal{R}](\hat{\eta})[K] = [K', \mathcal{R}](\hat{\eta}) = 0 \qquad \text{whenever} \quad K(u) = 0. \tag{6.29}$$

To establish the proposition, first note that

$$K' = K \equiv r\left(\frac{\partial^2}{\partial r^2} - \frac{\partial^2}{\partial u^2}\right) + [(\Gamma - 1)u + \Delta]\frac{\partial^2}{\partial u\partial r} + 2\frac{\partial}{\partial r} \tag{6.30}$$

for the operator $K(\Phi)$ in equation (6.23). Using the definition of $\mathcal{R}$ in equation (6.22) it is straightforward to verify that $[K', \mathcal{R}] = 0$, which establishes the proposition.

It is of interest to note the action of the recursion operator $\mathcal{R}$ on the symmetries $\hat{\eta}_1$, $\hat{\eta}_2$, $\hat{\eta}_4$, $\hat{\eta}_5$ and $\hat{\eta}_\Omega$ corresponding to the symmetry operators $Y_1$, $Y_2$, $Y_4$, $Y_5$ and $Y_\Omega$ discussed in equations (6.14) *et seq*. We find

$$\begin{aligned} &\mathcal{R}(\hat{\eta}_1) = (\Gamma - 1)\hat{\eta}_1 \qquad \mathcal{R}(\hat{\eta}_2) = 2(\Gamma - 1)\hat{\eta}_2 - \Gamma\Delta\hat{\eta}_1 \\ &\mathcal{R}(\hat{\eta}_4) = -\mathcal{R}^2(\hat{\eta}_5) \qquad \mathcal{R}(\hat{\eta}_5) = \hat{\eta}_4 \qquad \mathcal{R}(\hat{\eta}_\Omega) = \mathcal{R}(\Omega) \end{aligned} \tag{6.31}$$

where

$$\begin{aligned} &\hat{\eta}_1 = -1 \qquad \hat{\eta}_2 = u - \Delta \qquad \hat{\eta}_4 = -\mathcal{R}(\Phi) \\ &\hat{\eta}_5 = \Phi \qquad \hat{\eta}_\Omega = \Omega(u, r). \end{aligned} \tag{6.32}$$

Note that $\mathcal{R}(\Omega)$ and $\mathcal{R}(\Phi)$ both satisfy equation (6.10) if $\Omega$ and $\Phi$ satisfy the equation.

*6.2. The Lie–Bäcklund symmetries $Y_\Omega$ and $\hat{X}_\Omega$*

In this section we point out the connection between the symmetry $Y_\Omega$ associated with equation (6.10) and the Lie–Bäcklund symmetry $\hat{X}_\Omega$ (equation (5.47)) of the dispersionless TDNLS equations (4.3), where $\Omega(u,r) \equiv W^2(u,r)$ satisfies equation (6.10). Using equations (6.14)–(6.20) it follows that the dispersionless TDNLS equations admit the symmetry operator

$$\tilde{X}_\Omega = [(u-\Delta)\Omega_u - \Omega - r\Omega_r]\frac{\partial}{\partial x} + \Omega_u\frac{\partial}{\partial t} + \cdots \tag{6.33}$$

where $\theta$ is held constant. A straightforward calculation using equations (6.24)–(6.26) shows that the operator

$$X_\Omega = [(u-\Delta)\Omega_u - \Omega - r\Omega_r]\frac{\partial}{\partial x} + \Omega_u\frac{\partial}{\partial t} - r\Omega_r\theta_x\frac{\partial}{\partial \theta} + \cdots \tag{6.34}$$

is the generalization of $\tilde{X}_\Omega$ for $\theta \neq$ constant. The canonical Lie–Bäcklund operator $\hat{X}_{\Omega(u,r)}$ corresponding to the operator (6.34) is

$$\hat{X}_\Omega = D_x(Q^1_\Omega)\frac{\partial}{\partial u} + D_x[\Omega(u,r)v]\frac{\partial}{\partial v} + D_x[\Omega(u,r)w]\frac{\partial}{\partial w} + \cdots \tag{6.35}$$

where

$$Q^1_\Omega = \int_{r_0}^{r} r'\Omega_u(u,r')\,\mathrm{d}r' + \int_{u_0}^{u}\mathrm{d}u'\,[\Omega(u',r) + r\Omega_r(u',r) + [(\Gamma-1)u' + \Delta]\Omega_{u'}(u',r)]_{r_0} \tag{6.36}$$

which is the symmetry operator (5.47) with $\Omega \equiv W^2(u,r)$.

## 7. The symmetry algebra of the dispersionless TDNLS equations

The Lie point symmetries $X_1$, $X_2$, $h(\theta)X_3$, $X_4$ and $X_5$ admitted by the dispersionless TDNLS equations plus the Lie–Bäcklund symmetries (5.47) and (5.59) obtained in section 5 (see also section 6) define an infinite dimensional Lie algebra. Using the results (6.24)–(6.26) the canonical Lie–Bäcklund symmetry operators for these symmetries can be written in the form

$$\hat{X}_1 = -D_x(u)\frac{\partial}{\partial u} - D_x(r)\frac{\partial}{\partial r} - \theta_x\frac{\partial}{\partial\theta} + \cdots$$

$$\hat{X}_2 = D_x\left(\frac{\Gamma u^2 + r^2}{2}\right)\frac{\partial}{\partial u} + D_x[(u-\Delta)r]\frac{\partial}{\partial r} + (u-\Delta)\theta_x\frac{\partial}{\partial\theta}\ldots$$

$$h(\theta)\hat{X}_3 = h(\theta)\frac{\partial}{\partial\theta} + D_x[h(\theta)]\frac{\partial}{\partial\theta_x} + \cdots$$

$$\hat{X}_4 = D_x\{[(\Gamma-1)x - \Gamma\Delta t]u + \Delta x - (\Gamma-1)t(\Gamma u^2 + r^2)\}\frac{\partial}{\partial u}$$
$$+ D_x\{[(\Gamma-1)x - \Gamma\Delta t - 2(\Gamma-1)t(u-\Delta)]r\}\frac{\partial}{\partial r}$$

$$+[(\Gamma-1)x-\Gamma\Delta t-2(\Gamma-1)t(u-\Delta)]\theta_x\frac{\partial}{\partial\theta}+\cdots$$

$$\hat{X}_5=D_x[U-xu+\tfrac{1}{2}t(\Gamma u^2+r^2)]\frac{\partial}{\partial u}+D_x[R-xr+t(u-\Delta)r]\frac{\partial}{\partial r}$$

$$+[t(u-\Delta)-x]\theta_x\frac{\partial}{\partial\theta}+\cdots$$

$$\hat{X}_{p(\theta)}=\frac{p(\theta)\theta_x}{r}\frac{\partial}{\partial\theta}+D_x\left(\frac{p(\theta)\theta_x}{r}\right)\frac{\partial}{\partial\theta_x}+\cdots$$

$$\hat{X}_{\Omega}=D_x(Q^1_{\Omega})\frac{\partial}{\partial u}+D_x[r\Omega(u,r)]\frac{\partial}{\partial r}+\Omega(u,r)\theta_x\frac{\partial}{\partial\theta}+\cdots \tag{7.1}$$

where

$$U=D_x^{-1}(u)\qquad R=D_x^{-1}(r) \tag{7.2}$$

and $D_x = \partial/\partial x$ and $D_t = \partial/\partial t$. The potential $Q^1_{\Omega}$ in equation (7.1) is defined in equation (6.36). The operators (7.1) can be written in terms of the potential operators (equations (5.33)–(5.35)) if the canonical variables $(u, v, w)$ are used instead of $(u, r, \theta)$ as the basic variables.

Using the standard result

$$[\hat{X}(\hat{\eta}_1),\hat{X}(\hat{\eta}_2)]=\hat{X}(\hat{\eta}_3)\qquad \hat{\eta}_3=\hat{X}(\hat{\eta}_1)\hat{\eta}_2-\hat{X}(\hat{\eta}_2)\hat{\eta}_1 \tag{7.3}$$

for the commutator of two canonical Lie–Bäcklund symmetry operators [31], the symmetry algebra generated by the operators (7.1) has non-zero commutators

$$\begin{aligned}
&[\hat{X}_1,\hat{X}_4]=-(\Gamma-1)\hat{X}_1\qquad [\hat{X}_1,\hat{X}_5]=\hat{X}_1\\
&[\hat{X}_2,\hat{X}_4]=\Gamma\Delta\hat{X}_1-2(\Gamma-1)\hat{X}_2\qquad [\hat{X}_2,\hat{X}_5]=\hat{X}_2\\
&[\hat{X}_3,h(\theta)\hat{X}_3]=h'(\theta)\hat{X}_3\qquad [\hat{X}_3,\hat{X}_{p(\theta)}]=\hat{X}_{p'(\theta)}\\
&[h_1(\theta)\hat{X}_3,h_2(\theta)\hat{X}_3]=[h_1(\theta)h_2'(\theta)-h_2(\theta)h_1'(\theta)]\hat{X}_3\\
&[h(\theta)\hat{X}_3,\hat{X}_{p(\theta)}]=\hat{X}_{P(\theta)}\qquad P(\theta)=h(\theta)p'(\theta)-p(\theta)h'(\theta)\\
&[\hat{X}_4,\hat{X}_{\Omega(u,r)}]=\hat{X}_{\Lambda(u,r)}\qquad \Lambda(u,r)=\mathcal{R}[\Omega(u,r)]\\
&[\hat{X}_5,\hat{X}_{p(\theta)}]=-\hat{X}_{p(\theta)}\qquad [\hat{X}_5,\hat{X}_{\Omega}]=-\hat{X}_{\Omega}
\end{aligned} \tag{7.4}$$

plus the reverse commutators. In equations (7.4) $\Lambda(u,r)\equiv\mathcal{R}\Omega(u,r)$, and the recursion operator $\mathcal{R}$ are defined in equations (6.22). Both $\Omega(u,r)$ and $\Lambda(u,r)$ satisfy the linear partial differential equation (6.10). The commutation relations (7.4) show that the operators form a closed Lie algebra.

## 8. Concluding remarks

In this paper we have obtained infinite classes of Lie–Bäcklund symmetries and conservation laws for the dispersionless TDNLS equations describing the interaction of the Alfvén and magnetoacoustic waves near the triple umbilic point. At the triple point the gas sound speed $a_{\mathrm{g}}$ and Alfvén speed $V_{\mathrm{A}}$ coincide, and the Alfvén and the fast and slow magnetoacoustic speeds have the the same value for wave propagation along the magnetic field.

In section 2 an overview of the Lie point symmetries, Hamiltonian and Lagrangian variational formulations of the dispersive TDNLS equations was given. The dispersionless TDNLS equations are of hydrodynamic type; they have three families of characteristics analogous to the slow, fast and intermediate modes of MHD, and can be written in diagonal form in terms of the Riemann invariants (section 3).

Conservation laws for the dispersionless equations were obtained by three different methods. The first method (section 4) is to search for conserved densities $A(u, v, w)$ and fluxes $F(u, v, w)$, where $u$, $v$ and $w$ are the dependent variables by direct manipulation of the equations ($u$ represents the density perturbation, whereas $v$ and $w$ correspond to the transverse magnetic field perturbations). Two infinite families of conservation laws were obtained. The first family of conservation laws corresponds to conserved densities of the form $A = A(u, r)$ where $r = (v^2 + w^2)^{\frac{1}{2}}$. This class of conserved densities satisfies the linear second-order, hyperbolic differential equation (4.13) or (4.20), which admits separable solutions involving solutions of Legendre's equation (in some cases, the solutions are in terms of confluent hypergeometric functions, parabolic cylinder functions or modified Bessel functions depending on the values of the parameters). By appropriate choice of the separation parameter an infinite class of polynomial conserved densities in $u$ and $r$ are obtained. The second class of conserved densities are of the form $A = rh(\theta)$ where $\theta = \tan^{-1}(w/v)$ and $h(\theta)$ is an arbitrary differentiable function of $\theta$ (equation (4.15)).

Further insight into the symmetries and conservation laws of the dispersionless equations was obtained in section 5, by using the theory of symmetries of systems of hydrodynamic type developed by Dubrovin and Novikov [21], Tsarev [22, 23] and others (see, e.g., Ferapontov [24]) based on the diagonalized Riemann invariant form of the equations. It was shown that the dispersionless TDNLS equations are semi-Hamiltonian, implying the existence of an infinite class of conservation laws for the equations. The Riemann invariant analysis allows one to identify the Lie–Bäcklund symmetry associated with each conservation law. Since the TDNLS equations are Hamiltonian with symplectic operator $D_x$, it follows that the generators of the canonical Lie–Bäcklund symmetries are of the form $\hat{S}^\alpha = D_x(Q^\alpha)$ where the $Q^\alpha$ are potential operators [25]. The conserved densities $A$ can be reconstructed from the potential operators $Q^\alpha$ by the methods developed by Magri [25, 30].

The third method used to obtain Lie–Bäcklund symmetries was to note that the dispersionless equations, with $\theta$ taken as constant consist of two coupled equations of hydrodynamic type for $u(x, t)$ and $r(x, t)$ which can be linearized by the hodograph transformation. By writing $t(u, r) = \Phi_u(u, r)$, the hodograph equations for $x$ and $t$ can be formally integrated, where the function $\Phi(u, r)$ satisfies the second-order, linear, hyperbolic differential equation (6.10), which also arises in the Riemann invariant analysis of section 5. By determining the Lie point symmetries of the equation for $\Phi(u, r)$ and exploiting the hodograph transformation again leads to the infinite class of symmetries and conserved densities of the form $A = A(u, r)$ obtained in sections 4 and 5. The method also reveals a recursion operator for symmetries of the $\Phi$ equation (6.10). This development shows the central role of the hodograph transformation for the dispersionless TDNLS equations.

An overview of the symmetry algebra of the dispersionless TDNLS equations was provided in section 7.

A question not addressed by the present paper is the possibility of multi-Hamiltonian formulations of the dispersionless TDNLS equations. For $\Gamma = 1$ and $\Delta = 0$, the simplified dispersionless equations (6.3) may be identified as the equations of adiabatic compressible gas dynamics in one Cartesian space dimension with polytropic index $\gamma_{\mathrm{g}} = 3$, which are known to possess a multi-Hamiltonian formulation (e.g. [33]). Also of interest is the relation between the present results, and work by Zajaczkowski [34, 35] in magnetohydrodynamics using the generalized method of characteristics.

Studies of the dispersive and dissipative form of the TDNLS equations [20, 16] and related equations [36], show that sufficiently compressive solutions tend to develop shocks. This suggests that a study of the Riemann problem for the non-dispersive TDNLS equations should yield further insight into the formation of intermediate shocks, a subject of current interest in interplanetary physics [37].

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## Appendix

In this appendix we indicate the connection between the conserved densities $A_1$ and $A_2$ in equations (4.30) associated with space translation and time translation invariance and the polynomial conserved densities (4.28). Using the result

$$P_n^{(\alpha,\beta)}(x) = {}^{n+\alpha}C_n F(-n, n+\alpha+\beta+1; \alpha+1; \tfrac{1}{2}(1-x)) \tag{A.1}$$

relating the Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$ and the hypergeometric function $F(a, b; c; \frac{1}{2}(1-x))$ [28, formula 22.5.42, p 779], we find

$$\begin{aligned} a_0 &= 1 \qquad a_1 = (2-\Gamma)^{\frac{1}{2}}\left(u + \frac{\Delta}{\Gamma-1}\right) \\ a_2 &= r^2 + \left(u + \frac{\Delta}{\Gamma-1}\right)^2 \\ a_3 &= (2-\Gamma)^{\frac{1}{2}}\left[r^2\left(u + \frac{\Delta}{\Gamma-1}\right) + \frac{\Gamma}{3}\left(u + \frac{\Delta}{\Gamma-1}\right)^3\right] \end{aligned} \tag{A.2}$$

for the first few polynomial conserved densities (4.28) (set $\alpha_1 = 1$ and $\beta_1 = 1$ in equations (4.28)). Using the results (A.2) the expansions

$$\begin{aligned} A_1 &= \frac{1}{2}a_2 - \frac{\Delta}{(\Gamma-1)(2-\Gamma)^{\frac{1}{2}}}a_1 + \frac{\Delta^2}{2(\Gamma-1)^2}a_0 \\ A_2 &= \frac{1}{2}\left[\frac{a_3}{(2-\Gamma)^{\frac{1}{2}}} - \frac{\Gamma\Delta}{\Gamma-1}a_2 + \frac{\Gamma\Delta^2}{(\Gamma-1)^2(2-\Gamma)^{\frac{1}{2}}}a_1 - \frac{\Gamma\Delta^3}{3(\Gamma-1)^3}a_0\right] \end{aligned} \tag{A.3}$$

for the conserved densities $A_1$ and $A_2$ can be obtained straightforwardly.

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